LETTER FROM THE EDITOR

In mathematics, it is always satisfying to discover surprising connections among seemingly unrelated concepts. Our lead article is a marvelous illustration of this idea. Lagrange's identity is a famous algebraic result that involves products whose factors are each the sum of squares. Tommy Wright shows that combining this identity with a lot of ingenuity unites mathematical results like the Cauchy–Schwarz inequality, statistical concepts like the Pearson correlation coefficient inequality and the Neyman allocation scheme, and practical concerns like proportionate representation in the United States House of Representatives. Dr. Wright delivered one of the keynote addresses at the 2021 edition of MathFest, and I am very happy to help bring his insightful remarks to a wider audience.

Our co-lead article also has an applied bent. Robert A. Agnew considers the problem of income inequality in the United States. A common measure of inequality used by mathematicians and economists is the Gini index. Roughly, we might imagine that a perfectly equitable society is one in which the lowest p percent of income earners controls p percent of the total national income. This could be represented as a linear relationship on a graph. Modern society deviates from this relationship in the sense that rich people control more wealth, and poor people control less, than their proportions in the population suggest. On a graph, this relationship would be represented by an increasing, convex, curve that sits below the equity line. The area between these curves can be viewed as a measure of inequality, meaning that elementary calculus can be brought to the discussion. Agnew explores the relationship of different tax schemes to the Gini index and makes a simple proposal for reducing inequality.

The shorter pieces have a little something for everyone. Jonathan Hoseana and Ryan Aziz, who contributed the cover image for this issue, turn our focus to number theory. They study the orbits of the generalized sum of remainders map and make some interesting discoveries along the way. Douglas J. Durian revisits the venerable topic of Tangrams, searching for polyabolos with four-fold symmetry. Never heard of a "polyabolo"? Neither had I before reading this fascinating article. Konstantinos Gaitanas contributes a very clever proof of the arithmetic mean-geometric mean inequality. His proof is exceedingly clever and appears to be new, so hopefully it will evade any claims of anticipation.

Terence Perciante discusses polycycloids, which are the curves formed by given points on the perimeters of polygons as they roll over flat surfaces. Ulrich Abel discusses Halphen's identity, a little-known formula addressing the nth derivatives of certain products of functions. Bennett Eisenberg finds connections between triangular numbers and the "optimal stopping problem," a probability puzzle that has received considerable online attention in recent years.

We round out the issue with Problems, Reviews, and a presentation of the 2020 Carl B. Allendoerfer Awards. Proofs Without Words will make a triumphant return in our December issue.

Jason Rosenhouse, Editor

Math. Mag. **94** (2021) 243. doi:10.1080/0025570X.2020.1953839

ARTICLES

From Cauchy–Schwarz to the House of Representatives: Applications of Lagrange's Identity

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Joseph-Louis Lagrange (1736–1813) developed an early interest in mathematics and physics, having been mostly self-taught in mathematics. He published results on the calculus of variations, calculus of probabilities, and the integration of differential equations, and also made contributions to number theory and mechanics. By 1761, at the age of 25, he was recognized as one of the greatest living mathematicians. We all know about Lagrange multipliers and the Lagrangian function when optimizing functions subject to constraints. In 1773, Lagrange wrote down what is now called Lagrange's identity [**1**,**10**,**20**]. It is a simple yet elegant result. Like Lagrange multipliers, Lagrange's identity is useful in optimizing functions. See O'Connor and Robertson [**15**] for more on Lagrange.

Among the first few pages of Lagrange's paper [**10**], and using his notation, we pull out a sequence of results revealing what is now known as Lagrange's identity. He assumes we are given six numbers x', y', z', x'', y'', and z'', which can be viewed as the $n = 3$ ordered pairs (x', x''), (y', y''), and (z', z''). He then sets

$$\xi = y'z'' - z'y'' \qquad \eta = z'x'' - x'z'' \qquad \zeta = x'y'' - y'x''.$$

Taking $a' = (x')^2 + (y')^2 + (z')^2$, $a'' = (x'')^2 + (y'')^2 + (z'')^2$, and $b = x'x'' + y'y'' + z'z''$, leads to

$$\alpha = a'a'' - b^2 = \left[(x')^2 + (y')^2 + (z')^2\right]\left[(x'')^2 + (y'')^2 + (z'')^2\right] - \left[x'x'' + y'y'' + z'z''\right]^2.$$

Finally, he notes that $\alpha = \xi^2 + \eta^2 + \zeta^2$ which leads directly to Lagrange's identity (for $n = 3$):

$$\left[(x')^2 + (y')^2 + (z')^2\right]\left[(x'')^2 + (y'')^2 + (z'')^2\right] - \left[x'x'' + y'y'' + z'z''\right]^2 = \left[y'z'' - z'y''\right]^2 + \left[z'x'' - x'z''\right]^2 + \left[x'y'' - y'x''\right]^2.$$

Surprisingly, we can derive several big results in mathematics and statistics directly using Lagrange's identity. Was Lagrange aware that his groupings of various functions of the six numbers, that is, a', a'', b, and α leading to $a'a'' - b^2 = (\alpha) = \xi^2 + \eta^2 + \zeta^2$,

Math. Mag. **94** (2021) 244–256. doi:10.1080/0025570X.2021.1951575 © Mathematical Association of America
MSC: Primary 00-02, Secondary 00A05
This article has been corrected with minor changes. These changes do not impact the academic content of the article.

could be used to derive such big results? It is more likely that others knew of Lagrange's identity. We share four such results and wonder what other results are possible when there are n pairs of numbers: the Cauchy-Schwarz inequality, the Pearson correlation coefficient inequality, the Neyman allocation in probability sampling, and the method of equal proportions for apportionment of the United States House of Representatives.

Before proceeding, we change notation and state the general form of Lagrange's Identity as it is usually presented in recent writing.

Lagrange's identity

For any two ordered pairs of real numbers (a_1, b_1) and (a_2, b_2), it is straightforward to show that

$$(a_1^2 + a_2^2)(b_1^2 + b_2^2) - (a_1 b_1 + a_2 b_2)^2 = (a_1 b_2 - a_2 b_1)^2.$$

For any three ordered pairs of real numbers (a_1, b_1), (a_2, b_2), and (a_3, b_3), we can also show that

$$\begin{aligned}(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) &- (a_1 b_1 + a_2 b_2 + a_3 b_3)^2 \\ &= (a_1 b_2 - a_2 b_1)^2 + (a_1 b_3 - a_3 b_1)^2 + (a_2 b_3 - a_3 b_2)^2.\end{aligned}$$

In general, for any n ordered pairs of real numbers $(a_1, b_1), \ldots, (a_n, b_n)$, *Lagrange's identity* is

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) - \left(\sum_{i=1}^{n} a_i b_i\right)^2 = \sum_{1 \le i < j \le n} (a_i b_j - a_j b_i)^2, \tag{1}$$

which follows easily by mathematical induction.

The Cauchy–Schwarz inequality

For any n ordered pairs of real numbers $(a_1, b_1), \ldots, (a_n, b_n)$, the *Cauchy–Schwarz inequality* [**3**, **19**, **20**] is

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right). \tag{2}$$

Cauchy presented (2) in 1821 [**3**]. Related work by Schwarz [**19**] was reported in 1885. Equation (2) follows immediately from Lagrange's identity because the right-hand side of equation (1) is nonnegative. Note that

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) - \left(\sum_{i=1}^{n} a_i b_i\right)^2 = \sum_{1 \le i < j \le n} (a_i b_j - a_j b_i)^2 \ge 0. \tag{3}$$

In equation (2), the expression on the right-hand side is always bounded below by the expression on the left-hand side. From the right-hand side of equation (3), we easily see equality in equation (2) when $a_i/b_i = a_j/b_j = c$ for some number c and for all (a_i, b_i) and (a_j, b_j). That is, we have equality in equation (2) when $a_i = cb_i$ for all i where $i = 1, 2, \ldots, n$.

This is also when the expression on the right-hand side of equation (2) attains its minimum. This minimum is given by the expression of the left-hand side of equation (2).

An alternate form of the Cauchy–Schwarz inequality is (taking square roots of both sides of equation (2))

$$\left|\sum_{i=1}^{n} a_i b_i\right| \leq \sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2}. \tag{4}$$

Given the vectors $\vec{a} = \langle a_1, a_2, \ldots, a_n\rangle$ and $\vec{b} = \langle b_1, b_2, \ldots, b_n\rangle$, we can show that the triangle inequality $\|\vec{a} + \vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$ results from an application of equation (2). Indeed, there are many applications of equation (2) as shown in [**20**].

Pearson correlation coefficient inequality

One of the most important statistical concepts is that of correlation between two variables x and y. For example, assume a particular class with $n = 30$ students. If we were to record for each student the total time spent studying before an exam x and the score on that exam y, and plot the thirty ordered pairs (x, y), we might see that as the x values increase, the y values also increase. When this occurs, we say that x is correlated with y. More specifically, x is "positively" correlated with y. On the other hand, for variables x^* and y^*, if as the x^* values increase we observe that the y^* values decrease, then we could also say that x^* and y^* are correlated. In this case x^* and y^* are "negatively" correlated. It is well-known that a correlation between x and y does not necessarily mean that an increase in x causes an increase (or decrease) in y.

To quantify the amount of correlation, let's focus on the Pearson correlation coefficient [**18**, **23**], which has countless scientific applications. Galton gets credit for the early concept of correlation [**5**, **6**], and Pearson gets credit for some of the concept's mathematical development [**16**, **17**]. Let $\bar{x}$ be the simple average of the x values $x_1, x_2, \ldots, x_n$, and $\bar{y}$ be the simple average of the y values $y_1, y_2, \ldots, y_n$.

The *Pearson correlation coefficient* of x and y, or of the points

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$$

is

$$\rho = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2 \sum_{i=1}^{n}(y_i - \bar{y})^2}}. \tag{5}$$

The quantity ρ satisfies the *Pearson correlation coefficient inequality* $-1 \leq \rho \leq 1$.

To see this, note that setting $a_i = x_i - \bar{x}$ and $b_i = y_i - \bar{y}$ in Lagrange's identity gives:

$$\begin{aligned}\rho^2 &= \frac{\left[\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})\right]^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2 \sum_{i=1}^{n}(y_i - \bar{y})^2}\\ &= \frac{\left[\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})\right]^2}{\left[\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})\right]^2 + \sum_{1\leq i<j\leq n}\left[(x_i - \bar{x})(y_j - \bar{y}) - (x_j - \bar{x})(y_i - \bar{y})\right]^2}.\end{aligned} \tag{6}$$

Hence, $\rho^2 \leq 1$, or equivalently, $|\rho| \leq 1$. This implies $-1 \leq \rho \leq 1$, as claimed.

The points $(x_1, y_1), \dots, (x_n, y_n)$, as well as the point $(\bar{x}, \bar{y})$, all lie on the same straight line if and only if

$$(x_i - \bar{x})(y_j - \bar{y}) = (x_j - \bar{x})(y_i - \bar{y}) \tag{7}$$

for all i and j. That is, the points $(x_1, y_1), \dots, (x_n, y_n)$, and $(\bar{x}, \bar{y})$, all lie on the same straight line if and only if the slope of the straight line between (x_i, y_i) and $(\bar{x}, \bar{y})$ is the same as the slope of the straight line between (x_j, y_j) and $(\bar{x}, \bar{y})$ for all i and j. Thus, we see that $\rho^2 = 1$, or equivalently

$$\sum_{1 \le i < j \le n} \left[(x_i - \bar{x})(y_j - \bar{y}) - (x_j - \bar{x})(y_i - \bar{y})\right]^2 = 0$$

in equation (6), if and only if all points (x_i, y_i) for all i lie on the same straight line $y = a_0 x + b_0$ for real numbers a_0 and b_0. If $\rho = 1$, then $a_0 > 0$. If $\rho = -1$, then $a_0 < 0$. The closer ρ^2 is to 0, the more the points fail to lie on a single straight line.

Hence, we see that ρ measures the strength of the *linear* relation between the variables x and y. The connection between ρ and points on a straight line is not at all obvious from equation (5). However, Lagrange's identity helps us see it clearly.

From equation (7), we have conditions when

$$\left[\sum_{i=1}^{n} (x_i - \bar{x})^2\right]\left[\sum_{i=1}^{n} (y_i - \bar{y})^2\right]$$

attains its minimum value. This value is

$$\left[\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})\right]^2.$$

As with the Cauchy–Schwarz inequality, this happens when

$$\frac{y_i - \bar{y}}{x_i - \bar{x}} = \frac{y_j - \bar{y}}{x_j - \bar{x}} = d$$

for all i and j and some d. That is, when $\rho = -1$ (its minimum value), or when $\rho = 1$ (its maximum value).

Neyman allocation in probability sampling

Imagine that University ABC is concerned about the time spent studying by its undergraduate students. It plans to take a probability sample of $n = 100$ students to estimate the average number of hours spent studying each week. Believing that the number of hours varies among the four classes (that is, among freshmen, sophomores, juniors, and seniors), it is desired that a sample of $n = 100$ contain some students from each of the four classes and to not leave this to chance. If one stratifies (that is, partitions) the population of all undergraduate students into four disjoint groups (or strata) and then independently selects a probability sample from each group, this approach guarantees an overall sample that is representative of the student population generally. With this approach, there is a question of how to allocate optimally the 100 students among the four different classes. An answer has been offered by Neyman [**13**], in a seminal paper on probability sampling. It has been recognized that the result itself was reported earlier, in 1923 [**22**]. We provide some general background and then use Lagrange's identity to derive the result known as "Neyman allocation."

Assume a stratified (partitioned) finite population of $N = N_1 + \cdots + N_H$ units into H strata where N_i (the number of units in stratum i) is known for $i = 1, 2, \ldots, H$.

Let Y_{ij} be the fixed value of interest for the jth unit in stratum i. We use the notation

$$\bar{Y}_i = \frac{\sum_{j=1}^{N_i} Y_{ij}}{N_i} \qquad \text{and} \qquad S_i^2 = \frac{\sum_{j=1}^{N_i} (Y_{ij} - \bar{Y}_i)^2}{N_i - 1}.$$

Visually, the general setup is shown in Table 1.

TABLE 1: The general setup for a Neyman allocation scheme.

Stratum 1	Stratum 2	$\cdots$	Stratum i	$\cdots$	Stratum H
$Y_{11}, \ldots, Y_{1N_1}$ $\bar{Y}_1 \quad S_1^2$	$Y_{21}, \ldots, Y_{2N_2}$ $\bar{Y}_2 \quad S_2^2$	$\cdots$	$Y_{i1}, \ldots, Y_{ij}, \ldots, Y_{iN_i}$ $\bar{Y}_i \quad S_i^2$	$\cdots$	$Y_{H1}, \ldots, Y_{HN_H}$ $\bar{Y}_H \quad S_H^2$

The unknown value of the population total of the Y_{ij} values is

$$T_Y = \sum_{i=1}^{H} \sum_{j=1}^{N_i} Y_{ij} = \sum_{i=1}^{H} N_i \bar{Y}_i. \tag{8}$$

To estimate the value of T_Y under a sampling design-based approach, select a stratified random sample of $n = \sum_{i=1}^{H} n_i$ units where the n_i units selected from stratum i provide the sample average $\bar{y}_i$ for stratum i. The n_i units selected from stratum i are selected randomly so that each possible subset of n_i units from the N_i units has the same chance of being selected. The selection of the n_i units from stratum i is independent of the selection of the n_j units from stratum j. The union of the $n_1, n_2, \ldots, n_H$ units is called a *stratified random sample*. Under stratified random sampling, we must have $n_i \geq 1$ for all i.

A natural unbiased estimator for T_Y is

$$\hat{T}_Y = \sum_{i=1}^{H} N_i \bar{y}_i \tag{9}$$

with well-known sampling variance [**4**, **11**, **13**]. We have

$$\operatorname{Var}(\hat{T}_Y) = \sum_{i=1}^{H} N_i^2 \left(\frac{N_i - n_i}{N_i} \right) \frac{S_i^2}{n_i} = \sum_{i=1}^{H} \frac{N_i^2 S_i^2}{n_i} - \sum_{i=1}^{H} N_i S_i^2. \tag{10}$$

We want $\operatorname{Var}(\hat{T}_Y)$ to be small because $\hat{T}_Y$ is a random variable whose value depends on the selected stratified random sample. From equation (10), and for fixed overall sample size n and fixed stratification plan (that is, partition), it is clear that the only control we have to minimize $\operatorname{Var}(\hat{T}_Y)$ is through our wise choice of $n_1, n_2, \ldots, n_H$. So we could ask, "How to allocate fixed n among the H strata?" The allocation of fixed n that minimizes $\operatorname{Var}(\hat{T}_Y)$ is known as *Neyman Allocation* [13] and is given by

$$n_i = \frac{N_i S_i}{\sum_{j=1}^{H} N_j S_j} (n), \tag{11}$$

for $i = 1, \ldots, H$.

Following Cochran [4] and Stuart [21], for fixed n and constraint $n = \sum_{i=1}^{H} n_i$, finding $n_1, \ldots, n_H$ to minimize

$$\mathrm{Var}(\hat{T}_Y) = \sum_{i=1}^{H} \frac{N_i^2 S_i^2}{n_i} - \sum_{i=1}^{H} N_i S_i^2$$

is equivalent to finding $n_1, \ldots, n_H$ to minimize

$$\left(\sum_{i=1}^{H} \frac{N_i^2 S_i^2}{n_i} \right) (n),$$

which can be written as

$$\left(\sum_{i=1}^{H} \frac{N_i^2 S_i^2}{n_i} \right) (n) = \left(\sum_{i=1}^{H} \frac{N_i^2 S_i^2}{n_i} \right) \left(\sum_{i=1}^{H} n_i \right). \tag{12}$$

By Lagrange's identity with $a_i = N_i S_i / \sqrt{n_i}$ and $b_i = \sqrt{n_i}$, we have that:

$$\left[\sum_{i=1}^{H} \left(\frac{N_i S_i}{\sqrt{n_i}} \right)^2 \right] \left[\sum_{i=1}^{H} (\sqrt{n_i})^2 \right] - \left[\sum_{i=1}^{H} \frac{N_i S_i}{\sqrt{n_i}} \sqrt{n_i} \right]^2$$
$$= \sum_{1 \le i < j \le H} \left(\frac{N_i S_i}{\sqrt{n_i}} \sqrt{n_j} - \frac{N_j S_j}{\sqrt{n_j}} \sqrt{n_i} \right)^2 \ge 0. \tag{13}$$

Clearly, this implies

$$\left[\sum_{i=1}^{H} \left(\frac{N_i S_i}{\sqrt{n_i}} \right)^2 \right] \left[\sum_{i=1}^{H} (\sqrt{n_i})^2 \right] \ge \left[\sum_{i=1}^{H} \frac{N_i S_i}{\sqrt{n_i}} \sqrt{n_i} \right]^2. \tag{14}$$

Applying Lagrange's identity, we have equality in equation (14) and achieve a minimum for equation (10) or (12) if and only if (from the right-hand side of equation (13))

$$\frac{\sqrt{n_i}}{\left(\frac{N_i S_i}{\sqrt{n_i}} \right)} = \frac{\sqrt{n_j}}{\left(\frac{N_j S_j}{\sqrt{n_j}} \right)} = k, \tag{15}$$

where k is a constant for all i and j. Hence the minimum value of $\mathrm{Var}(\hat{T}_Y)$ occurs when equation (15) holds. That is, when for some constant k, we have for all i that $n_i = k N_i S_i$

In other words, the left-hand side of equation (14), which we want to minimize, attains its minimum value if and only if equation (15) holds. This is equivalent to

$$\sum_{1 \le i < j \le H} \left(\frac{N_i S_i}{\sqrt{n_i}} \sqrt{n_j} - \frac{N_j S_j}{\sqrt{n_j}} \sqrt{n_i} \right)^2 = 0. \tag{16}$$

We now have

$$\frac{n_i}{n} = \frac{n_i}{\sum_{j=1}^{H} n_j} = \frac{k N_i S_i}{\sum_{j=1}^{H} k N_j S_j} = \frac{N_i S_i}{\sum_{j=1}^{H} N_j S_j}.$$

This leads to *Neyman Allocation*, for all i

$$n_i = \frac{N_i S_i}{\sum_{j=1}^{H} N_j S_j} n.$$

Because n_i is rarely an integer, rounding or other methods are needed [24, 25].

Equal proportions: United States House of Representatives

Given the impracticality of a pure democracy, the United States Constitution of 1787 calls for a representative form of democracy where the people elect representatives to govern. Each state gets two senators for the Senate and some number of representatives for the House of Representatives "... according to their respective numbers ... " as recorded in a census of the nation to be conducted every ten years starting in 1790.

Article 1, Section 2, Clause 3 of the Constitution says

> Representatives ... shall be apportioned among the several States... according to their respective Numbers ... The actual Enumeration ... every ... ten years ... The Number of Representatives ... each state shall have at least one Representative ... and until such enumeration shall be made, the State of New Hampshire shall be entitled to chuse three, Massachusetts eight, Rhode-Island and Providence Plantations one, Connecticut five, New-York six, New Jersey four, Pennsylvania eight, Delaware one, Maryland six, Virginia ten, North Carolina five, South Carolina five, and Georgia three.

With apportionment, we seek to distribute or allocate H seats in the House of Representatives among all states based on each state's share of the national population. If a state's share of the H seats is not a positive integer, what about the fractional part? Should we round up, down, or not at all? Fractional parts have raised concerns since 1790 [2]. The first apportionment of the seats in the House of Representatives involved distributing 65 seats among the 13 original states as given explicitly in the Constitution, as noted above.

The apportionment method used every ten years since the 1940 Census to redistribute $H = 435$ seats in the House of Representatives among the states is called the *method of equal proportions*. For five selected states, Table 2 compares each state's share of seats with the number of seats allocated by the method of equal proportions following the 2010 Census. In Table 2, the 2010 Census population for the entire country was 309,183,463 people of which 4,802,982 were in Alabama. Hence, Alabama's share of the 435 seats in the House of Representatives is 6.757 seats. The method of equal proportions gives Alabama seven of the 435 seats. We also see that Rhode Island's share of the 435 seats is 1.485, yet the method of equal proportions gives Rhode Island two seats. We derive the method of equal proportions using Lagrange's identity.

Preliminary to our derivation of the method of equal proportions, define the following notation:

$$P_i = \text{number of people in state } i,$$

$$H_i = \text{number of seats to be assigned to state } i,$$

$$P = \sum_i P_i = \text{ the total population of the states },$$

$$H = \sum_i H_i = \text{ the fixed total number of seats in the House of Representatives.}$$

TABLE 2: Number of seats allocated to five selected states following the 2010 Census based on the method of equal proportions.

State	Share of seats	Number of seats
Alabama	$\frac{4802982}{309183463}(435) \approx 6.757$	7
Maryland	$\frac{5789929}{309183463}(435) \approx 8.146$	8
North Carolina	$\frac{9565781}{309183463}(435) \approx 13.458$	13
Minnesota	$\frac{5314879}{309183463}(435) \approx 7.478$	8
Rhode Island	$\frac{1055247}{309183463}(435) \approx 1.485$	2

Building on the work of Hill [7], Huntington [8, 9] takes as a fundamental principle that "the state i value P_i/H_i should be as nearly equal as possible to the state j value P_j/H_j." Huntington's definition of "inequality" is relative difference. That is, we consider the ratio

$$\frac{\left|\frac{P_i}{H_i} - \frac{P_j}{H_j}\right|}{\min\left(\frac{P_i}{H_i}, \frac{P_j}{H_j}\right)}. \tag{17}$$

Huntington's method of equal proportions is presented as the 3-step algorithm

(1) Assign one representative (seat) to each state.

(2) Compute the array of *priority values*, assuming that $P_1 \geq P_2 \geq P_3 \geq \cdots \geq P_{50}$:

$$\begin{array}{lcccc}
\text{State } 1 & \frac{P_1}{\sqrt{1\cdot 2}} & \frac{P_1}{\sqrt{2\cdot 3}} & \frac{P_1}{\sqrt{3\cdot 4}} & \cdots \\
 & & \vdots & & \\
\text{State } i & \frac{P_i}{\sqrt{1\cdot 2}} & \frac{P_i}{\sqrt{2\cdot 3}} & \frac{P_i}{\sqrt{3\cdot 4}} & \cdots \\
 & & \vdots & & \\
\text{State } 50 & \frac{P_{50}}{\sqrt{1\cdot 2}} & \frac{P_{50}}{\sqrt{2\cdot 3}} & \frac{P_{50}}{\sqrt{3\cdot 4}} & \cdots
\end{array}$$

(3) Pick the $H - 50$ largest priority values with the associated states. Each state gets an additional seat each time one of its priority values is among the $H - 50$ largest values.

Huntington gives a proof of its optimality. We seek to give an alternative direct derivation which seems clearer and more insightful. Our alternative derivation builds on Wright [26]. Motivated by Huntington's formulation for fixed P_i and P_j, and by the constraint that all H_i should sum to the fixed value H, we seek to determine H_i and H_j so that $P_i/H_i = P_j/H_j$ for all i and j such that $i < j$. The equation implies that each representative should represent the same number of people among all states.

Equivalently, for all i and j such that $i < j$, the equation $P_i/H_i = P_j/H_j$ can be written as

$$\frac{P_i}{\sqrt{H_i}}\sqrt{H_j} - \frac{P_j}{\sqrt{H_j}}\sqrt{H_i} = 0. \tag{18}$$

Equality is rarely possible. Thinking about equation (18) leads us to determine H_i to minimize

$$\sum_{1\le i<j\le 50}\left(\frac{P_i}{\sqrt{H_i}}\sqrt{H_j}-\frac{P_j}{\sqrt{H_j}}\sqrt{H_i}\right)^2 \tag{19}$$

subject to the constraint $\sum_{i=1}^{50} H_i = H$. In Lagrange's identity, take

$$a_i = \frac{P_i}{\sqrt{H_i}}, \qquad b_i = \sqrt{H_i}, \qquad \text{and} \qquad n = 50,$$

thereby obtaining

$$\left(\sum_{i=1}^{50}\frac{P_i^2}{H_i}\right)\left(\sum_{i=1}^{50}H_i\right)-\left(\sum_{i=1}^{50}P_i\right)^2=\sum_{1\le i<j\le 50}\left(\frac{P_i}{\sqrt{H_i}}\sqrt{H_j}-\frac{P_j}{\sqrt{H_j}}\sqrt{H_i}\right)^2. \tag{20}$$

Proceeding as in [14], and for any positive integer H_i, we have that

$$\begin{aligned}1-\frac{1}{H_i}&=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{H_i-1}-\frac{1}{H_i}\right)\\&=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{(H_i-1)\cdot(H_i)}.\end{aligned}$$

Thus, we see that

$$\frac{1}{H_i}=1-\frac{1}{1\cdot 2}-\frac{1}{2\cdot 3}-\frac{1}{3\cdot 4}-\cdots-\frac{1}{(H_i-1)\cdot(H_i)}. \tag{21}$$

Substituting equation (21) into equation (20), we obtain

$$\begin{aligned}&\sum_{1\le i<j\le 50}\left(\frac{P_i}{\sqrt{H_i}}\sqrt{H_j}-\frac{P_j}{\sqrt{H_j}}\sqrt{H_i}\right)^2\\&=\left(\sum_{i=1}^{50}\frac{P_i^2}{H_i}\right)\left(\sum_{i=1}^{50}H_i\right)-\left(\sum_{i=1}^{50}P_i\right)^2\\&=\left[\sum_{i=1}^{50}P_i^2\left(\frac{1}{H_i}\right)\right]H-P^2\\&=\left[\sum_{i=1}^{50}P_i^2\left(1-\frac{1}{1\cdot 2}-\frac{1}{2\cdot 3}-\cdots-\frac{1}{(H_i-1)(H_i)}\right)\right]H-P^2\\&=H\sum_{i=1}^{50}P_i^2-P^2-H\sum_{i=1}^{50}\left(\frac{P_i^2}{1\cdot 2}+\frac{P_i^2}{2\cdot 3}+\cdots+\frac{P_i^2}{(H_i-1)(H_i)}\right).\end{aligned} \tag{22}$$

That is,

$$\sum_{1\le i<j\le 50}\left(\frac{P_i}{\sqrt{H_i}}\sqrt{H_j}-\frac{P_j}{\sqrt{H_j}}\sqrt{H_i}\right)^2$$
$$\begin{aligned}
&=\left(H\sum_{i=1}^{50}P_i^2-P^2\right)-\frac{HP_1^2}{1\cdot 2}-\frac{HP_1^2}{2\cdot 3}-\cdots-\frac{HP_1^2}{(H_1-1)(H_1)}\\
&\qquad-\frac{HP_2^2}{1\cdot 2}-\frac{HP_2^2}{2\cdot 3}-\cdots-\frac{HP_2^2}{(H_2-1)(H_2)}\\
&\qquad\vdots\\
&\qquad-\frac{HP_i^2}{1\cdot 2}-\frac{HP_i^2}{2\cdot 3}-\cdots-\frac{HP_i^2}{(H_i-1)(H_i)}\\
&\qquad\vdots\\
&\qquad-\frac{HP_{50}^2}{1\cdot 2}-\frac{HP_{50}^2}{2\cdot 3}-\cdots-\frac{HP_{50}^2}{(H_{50}-1)(H_{50})}.
\end{aligned}\tag{23}$$

The terms being subtracted on the ith row of equation (23) correspond to the ith state.

From equation (22), note that the amount by which

$$\sum_{1\le i<j\le 50}\left(\frac{P_i}{\sqrt{H_i}}\sqrt{H_j}-\frac{P_j}{\sqrt{H_j}}\sqrt{H_i}\right)^2$$

decreases when we increase the number of representatives in state i from H_i-1 to H_i is given by

$$\left\{\left[P_i^2\left(\frac{1}{H_i-1}\right)\right]H\right\}-\left\{\left[P_i^2\left(\frac{1}{H_i}\right)\right]H\right\}=\frac{P_i^2H}{(H_i-1)(H_i)}.\tag{24}$$

Also, from equations (22) and (23), note that

$$\left(H\sum_{i=1}^{50}P_i^2-P^2\right)$$

is fixed and that it is the value of

$$\sum_{1\le i<j\le 50}\left(\frac{P_i}{\sqrt{H_i}}\sqrt{H_j}-\frac{P_j}{\sqrt{H_j}}\sqrt{H_i}\right)^2$$

when $H_i=1$ for all i (from equation (22)). It is also the maximum value of equation (19).

Finally, we consider the sum of squares and the decomposition of equation (19) given in equation (23) and conclude the following: We minimize equation (19) by picking the largest terms subtracted in the overall sum, subject to the constraint. That is, minimization of equation (19) is attained when

(1′) Each state is assigned one seat in the House of Representatives, and

(2′) each state receives an additional seat each time it has a term which appears among the $H-50$ largest terms subtracted in equation (23).

Clearly, (1′) corresponds to (1) in the equal proportions algorithm presented earlier. To see that (2′) above corresponds to (2) and (3) of the algorithm, we make two comments. First, because each of the terms being subtracted on the right-hand side of equation (23) has H as a factor, H is ignored in our earlier presentation of the algorithm in terms of the priority values, for simplicity. Also, because the terms being subtracted are all squares of positive real numbers, the ordering of these terms is unchanged if we consider square roots instead in our earlier presentation of the algorithm. Thus, (1′) and (2′) are equivalent to the equal proportions algorithm.

The derivation for the method of equal proportions is complete.

Example of equal proportions optimality Huntington [**8**,**9**] proves that the method of equal proportions minimizes inequality (that is, it minimizes relative difference in the sense of equation (17) between any two states. For example, and using the P_i from the 2010 Census, Table 3 shows that the method of equal proportions gives 11 seats to Virginia and 10 seats to Washington in the House of Representatives. The relative difference for these seat assignments is 0.08198. If we transfer one seat from Virginia to Washington (Plan A), the relative difference is 0.30920, which is higher than what is given by the method of equal proportions. Similarly, if we transfer one seat from Washington to Virginia (Plan B), the relative difference is 0.12028, also higher than what is given by the method of equal proportions. The same holds for any pair of states with Huntington's method of equal proportions. With the method of equal proportions, the relative difference can not be made smaller by the transfer of one seat from any one state to another.

TABLE 3: Illustration that equal proportions minimizes inequality between two states.

State i	P_i	Equal proportions	Plan A	Plan B
Virginia	8,037,736	11	10	12
Washington	6,753,369	10	11	9

Concluding remarks

We presented elementary derivations of four well-known results using Lagrange's identity. This shows just how closely related to the results are

I. Cauchy–Schwarz Inequality
II. Pearson Correlation Coefficient Inequality
III. Neyman Allocation in Probability Sampling
IV. Equal Proportions: U.S. House of Representatives

Remark 1 Our derivations for I, II, and III are very similar. In each case, $\left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right)$ is shown to attain its minimum value, which is $\left(\sum_{i=1}^n a_i b_i\right)^2$ when $\sum_{1\le i<j\le n}(a_i b_j - a_j b_i)^2 = 0$.

Remark 2 In all four derivations, we want $\sum_{1\le i<j\le n}(a_i b_j - a_j b_i)^2 = 0$, or equivalently $\frac{a_i}{b_i} = \frac{a_j}{b_j} = c$ for all i and j and some constant c. In derivation II, we have "perfect" linear correlation. That is, $\rho^2 = 1$ when the sum equals 0.

Remark 3 The original proof of the Cauchy–Schwarz inequality is given as Theorem XVI of Note II of Cauchy [3]. It actually makes use of what is now known as Lagrange's identity, though there is no explicit reference to Lagrange. In fact, given that Lagrange (1736–1813) and Cauchy (1789–1857) overlapped each other by a few years, as well as the immediate connection between Lagrange's identity and the Cauchy–Schwarz inequality, it would not be surprising to learn that Lagrange had written down the Cauchy–Schwarz Inequality himself!

Remark 4 Wright [24] observes that the problem of allocating a sample among the strata and the problem of allocating seats in the House of Representatives among the states are both special cases of a more general problem, without mention of connections to Lagrange's Identity.

Remark 5 When $n = 2$ in Lagrange's identity, we are able to view the area of a parallelogram computed in two different ways. First, let $A(\vec{a}, \vec{b})$ be the size of the angle between the two vectors $\vec{a} = \langle a_1, a_2\rangle$ and $\vec{b} = \langle b_1, b_2\rangle$, and let $\det(\vec{a}, \vec{b}) = a_1b_2 - a_2b_1$ be the determinant of the 2×2 matrix with $\vec{a}$ in the first row and $\vec{b}$ in the second. Also, the dot product of $\vec{a}$ and $\vec{b}$ is $\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\| \cos[A(\vec{a}, \vec{b})] = a_1b_1 + a_2b_2$. It is known that the area of the parallelogram formed by $\vec{a}$ and $\vec{b}$ is given by $\det(\vec{a}, \vec{b})$, as well as by $\|\vec{a}\|\|\vec{b}\| \sin[A(\vec{a}, \vec{b})]$. Using Lagrange's identify yields

$$\begin{aligned}
\|\vec{a}\|^2\|\vec{b}\|^2 \sin^2\left[A(\vec{a}, \vec{b})\right] &= \|\vec{a}\|^2\|\vec{b}\|^2\left[1 - \cos^2\left[A(\vec{a}, \vec{b})\right]\right] \\
&= \|\vec{a}\|^2\|\vec{b}\|^2 - \|\vec{a}\|^2\|\vec{b}\|^2 \cos^2\left[A(\vec{a}, \vec{b})\right] \\
&= (a_1^2 + a_2^2)(b_1^2 + b_2^2) - (a_1b_1 + a_2b_2)^2 \\
&= (a_1b_2 - a_2b_1)^2 = \left[\det(\vec{a}, \vec{b})\right]^2.
\end{aligned}$$

Taking square roots demonstrates the equivalence of the two views for computing the area of a parallelogram. That is, Lagrange's identity connects the two views and reveals the equivalence.

Acknowledgments The views expressed in this article are those of the author and not those of the U.S. Bureau of the Census. The author is grateful to the referees whose comments greatly improved the presentation and content, especially the title and Remark 5.

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Summary. The simple and elegant result known as Lagrange's identity can be used to derive some well-known results in mathematics and statistics. We share four such results that have elementary derivations based on this identity: the Cauchy–Schwarz inequality, the Pearson correlation coefficient inequality, the Neyman allocation in probability sampling, and the method of equal proportions for apportionment of the United States House of Representatives.

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Income Inequality and Taxation

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It is hard to pick up a magazine these days without encountering an extensive article on income inequality, wealth inequality or both. Moreover, acclaimed books by economists Piketty [9], Reich [10], Stiglitz [13], and most recently Saez and Zuchman [11] have highlighted the issue for the general public, while Democratic presidential candidates have highlighted comparisons between the top 1% and the bottom 99%. There is no doubt that income inequality is a serious economic and political issue, particularly in the United States. The purpose of this article is to indicate, using straightforward mathematics, how income inequality can be reduced significantly via a simple linear income tax and to illustrate this with data from the United States Census Bureau and Internal Revenue Service (IRS).

Fontenot, Semega, and Kollar [4], in Table 2, present two traditional measures of income inequality:

(1) Shares of aggregate household income received by population quintiles (and the top 5%).

(2) The Gini index, plus the Theil index (similar to Gini) and other auxiliary metrics.

The Gini index attempts to capture distributional income inequality in one single number. Some, including Piketty, take issue with this view, but we focus entirely on that particular measure.

In 2017, per Census, 22.3% of the U.S. household income went to the top 5% of income-earners, while only 25.6% went to the bottom 60% of income-earners. The U.S. Gini index was 0.482, which Stiglitz notes is high relative to other developed nations and significantly increased from 1980. The IRS does not focus on inequality, but analysis of their tabulated data indicates that our current complex tax system does little to alleviate it. Indeed, average tax rates are not progressively higher at the top end of the adjusted gross income scale. Once again, these figures highlight the extent of the problem in the U.S.

Mathematicians were recently introduced to the Gini index, and various approximations to it, by Farris [3] and Jantzen and Volpert [6]. These authors have demonstrated that there is some interesting mathematics associated with income inequality measurement. Moreover, Farris developed a useful interpolation scheme for stratified income tables that we utilize later.

The next section reviews the Gini index, both pre-tax and post-tax, in general terms. We then get more specific with the linear income tax, much studied by economists [1, 8, 12], as well as in more recent economic reviews of optimal taxation [2, 7]. We derive the relationship between pre-tax and post-tax Gini, which indicates that significant inequality reduction is possible with a very simple linear tax structure. In the final section, we benchmark against actual U.S. income data from the Census Bureau [14] and the IRS [15].

Math. Mag. **94** (2021) 257–266. doi:10.1080/0025570X.2021.1957393
MSC: 91B15

Income inequality measurement: The Gini index

We start with the Lorenz curve $L(p)$ which defines the cumulative share of total money income associated with the lowest-income proportion p of a population. In other words, the cumulative fraction p of the population gets cumulative fraction $L(p)$ of total money income, arranged in order of increasing money income rate. To illustrate this concept, we already saw from 2017 Census data that $L(.6) = 0.256$ and $L(.95) = 1 - .223 = 0.777$. Mathematically, the Lorenz curve is a nondecreasing, convex function from [0, 1] onto itself with $L(0) = 0$ and $L(1) = 1$, while we normally expect the curve to be strictly increasing and strictly convex throughout. In this context, $L'(p)$ can be viewed as either a probability density function on the unit interval, or alternatively as the scaled money income rate at the lowest-income proportion p of the population. We will take the latter view. We assume throughout that we can treat L as twice continuously differentiable on the unit interval, notwithstanding the inherent discreteness in any real population.

From Farris [3], the Gini index of income inequality is defined by

$$G = 2 \int_0^1 (p - L(p))\, dp \tag{1}$$

The Gini index is portrayed graphically in Figure 1 as twice the area between the equity reference line and the Lorenz curve.

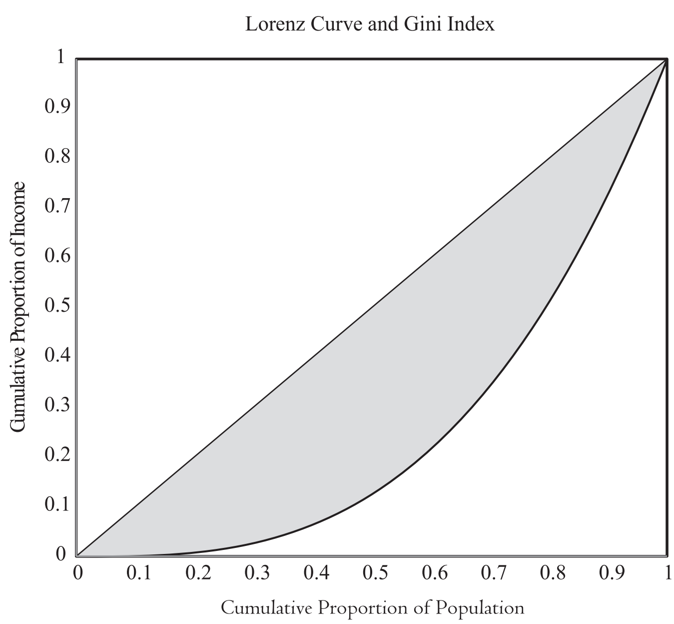

Figure 1 The Gini index is twice the area between equity reference line and the Lorenz curve.

A Gini of zero indicates perfect equality (per the equity reference line), whereas a Gini of one indicates perfect inequality in a limiting sense (per the boundary lines). As indicated previously, various approximations to both the Lorenz curve and the associated Gini index have been suggested, but for our derivation we simply assume that the Lorenz curve is given and we do not worry about its exact functional form.

We now introduce taxation as follows: Relabel the pre-tax Lorenz curve as L_0 and its associated Gini index as G_0. We introduce the cumulative tax curve as $F(p)$, the cumulative proportion of pre-tax money income received as tax from the lowest-income proportion p of the population. Then $F(0) = 0$ and $F(1) = c$, where $c \in (0, 1)$ represents the total proportion of pre-tax money income collected by the government. In this setup, $F'(p)$ is the scaled monetary tax rate at the lowest-income proportion p and $F'(p)/L_0'(p)$ is the percentage tax rate at p. For simplicity, our focus is entirely on personal or individual income taxation at the federal level. That is, we ignore corporate income taxes, all state and local taxes, and all other federal taxes.

We expect F' to be increasing (i.e., $F'' > 0$) throughout, but we need $F'(p)/L_0'(p)$ to be increasing throughout for a progressive percentage tax rate schedule. We can certainly have $F(p) < 0$ and $F'(p) < 0$ for lower portions of the curve, corresponding to a negative income tax which subsidizes lower-income individuals and is fundamental to significantly reducing inequality. It is worth noting that the negative income tax was favored by Milton Friedman [**5**] as a means of alleviating poverty on the lower end of the income scale.

We now define $H(p) = L_0(p) - F(p)$ as cumulative post-tax income as a proportion of total pre-tax income. We understand that the government is a large-scale employer, but there is no inconsistency if pre-tax income of government workers is included in L_0. We then have $H(0) = 0$, $H(1) = 1 - c$, and we must have both $H' > 0$ and $H'' > 0$ to make sense. The basic curves are illustrated schematically in Figure 2.

We now define the post-tax Lorenz curve as

$$L_1(p) = \frac{L_0(p) - F(p)}{1 - c}$$

so that $L_1(0) = 0$ and $L_1(1) = 1$. Then the post-tax Gini index is defined by

$$\begin{aligned} G_1 &= 2\int_0^1 (p - L_1(p))\,dp \\ &= 2\int_0^1 \left(p - \frac{L_0(p) - F(p)}{1 - c}\right) dp \\ &= G_0 - \frac{2\int_0^1 (cL_0(p) - F(p))\,dp}{1 - c} \end{aligned} \tag{2}$$

so that Gini reduction is proportional to the area between two cumulative curves, $cL_0(p)$ and $F(p)$, both joining points $(0, 0)$ and $(1, c)$ (see Figure 2). The curve $cL_0(p)$ is c times the solid line (plotted as dash-dot), and $F(p)$ is the dotted line. The curve $cL_0(p)$ is the cumulative tax curve associated with a non-progressive “flat” tax with constant tax rate $F'(p)/L_0'(p) = c$ across the whole income spectrum. Hence, from (2), $G_1 = G_0$ for a flat tax and there is no Gini reduction. We will investigate this relationship further in the next section for a full-fledged linear income tax. Figure 3 depicts a progressive tax schedule yielding significant Gini reduction.

Linear income tax

The linear income tax is defined by

$$F'(p) = rL_0'(p) - s$$

where $r \in (0, 1)$ is the marginal percentage tax rate and s is a common scaled monetary subsidy that everybody gets, often called “universal basic income.” In this context, the

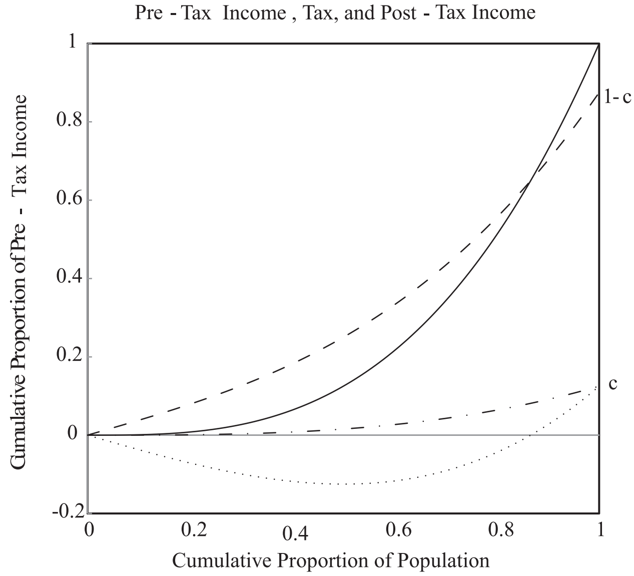

Figure 2 The solid line is pre tax income. Tax is dotted, post-tax income is dashed, and the flat-tax dash-dot, all proportional to pre-tax income. The tax curve can be negative on the low end. The Gini reduction is proportional to the area between the flat tax and tax curves.

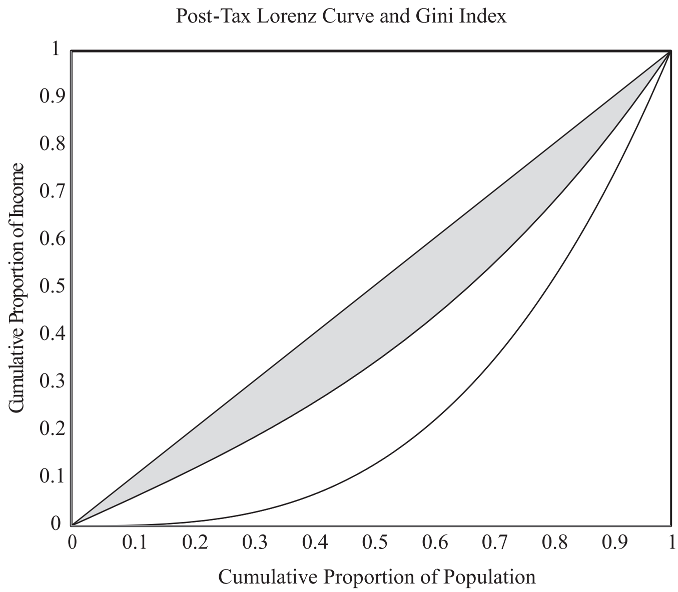

Figure 3 Post-tax Gini index is twice the area between the equity reference line and the post-tax Lorenz curve. The pre-tax Lorenz curve is also shown.

tax schedule is progressive if $F'(p)/L_0'(p)$ is increasing, which is true if $s > 0$. Upon integrating, we get $F(p) = rL_0(p) - sp$. Since $F(1) = c$, we have that $c = r - s$ or $s = r - c$. Hence, the cumulative tax curve is

$$F(p) = rL_0(p) - (r - c)p \tag{3}$$

and progressivity requires that $r > c$. Moreover, if

$$rL_0'(0) < r - c$$

we will have negative taxation on the lower end of the income scale. If $r = c$, we have the non-progressive flat tax referenced in the last section. If $r < c$, we have a regressive tax schedule with negative subsidy.

The general linear income tax has been studied extensively by economists in the context of both fair distribution and economic efficiency [**1**, **8**, **12**]. In particular, Aumann and Kurz [**1**] derived it via the Shapley value of cooperative game theory in the context of a majority–minority political game with threats. Coming from somewhat different directions, all these economists arrive at a version of (3) with positive subsidies and marginal tax rates in the 50%+ range, depending on utility assumptions. For linear monetary utility, Aumann and Kurz get exactly $r = 50\%$ with higher marginal tax rates for more concave monetary utility functions. Now we turn to calculation of post-tax Gini for the general linear income tax. We prove the following proposition:

Proposition. *For pre-tax Lorenz curve L_0, associated Gini index G_0, and cumulative tax curve defined by (3), the post-tax Gini index is*

$$G_1 = \left(\frac{1-r}{1-c}\right) G_0 \tag{4}$$

with percentage Gini reduction

$$1 - \frac{G_1}{G_0} = \frac{r-c}{1-c} \tag{5}$$

Proof. Using (3), we have

$$\begin{aligned} 2\int_0^1 (cL_0(p) - F(p))\,dp &= 2\int_0^1 (cL_0(p) - rL_0(p) + (r-c)p)\,dp \\ &= (r-c)G_0. \end{aligned}$$

Hence from (2),

$$G_1 = G_0 - \left(\frac{r-c}{1-c}\right) G_0 = \left(\frac{1-r}{1-c}\right) G_0,$$

and further,

$$1 - \frac{G_1}{G_0} = \frac{r-c}{1-c}.$$

■

Under a linear income tax structure, we conclude that inequality reduction is a function of the relationship between r and c. If r is much larger than c, then significant inequality reduction is possible. In the next section, we refer to empirical data from the U.S. Census Bureau and the U.S. Internal Revenue Service.

Benchmarking against the U.S. income data

So far our development has been continuous and fairly abstract. We now want to benchmark against actual tabulated U.S. income data, from both the Census Bureau and the IRS. In the interest of space, we will not reproduce their tables here. Instead, we will simply present highlights. To begin, we adopt the tabular notation of Farris [3] where we have n household strata arranged from lowest to highest average income with the following definitions:

$$h_i = \text{Number of Households in Stratum } i$$

$$N = \sum_{i=1}^{n} h_i = \text{Total Number of Households}$$

$$p_j = \sum_{i=1}^{j} h_i/N = \text{Cumulative Household Proportion through Stratum } j$$

$$x_i = \text{Average U.S. Dollar Annual Pre-Tax Money Income per Household in Stratum } i$$

$$T = \sum_{i=1}^{n} x_i h_i = \text{Total Household Dollar Income}$$

$$\bar{x} = T/N = \text{Overall Average Dollar Income}$$

From these ingredients, Farris noted that the pre-tax Lorenz curve

$$L_0(p_j) = \sum_{i=1}^{j} \frac{x_i h_i}{T}$$

through stratum j, and then he derived the following interpolation formula for the pre-tax Gini index:

$$G_0 \cong \left(\frac{1}{T} \sum_{j=1}^{n} x_j (p_j + p_{j-1}) h_j \right) - 1 \tag{6}$$

where $p_0 = 0$. An equivalent interpolation formula is

$$\begin{aligned} G_0 &\cong 1 - 2\sum_{j=1}^{n} \int_{p_{j-1}}^{p_j} \left(L_0(p_{j-1}) + \left(\frac{L_0(p_j) - L_0(p_{j-1})}{p_j - p_{j-1}} \right) (p - p_{j-1}) \right) dp \\ &= 1 - \sum_{j=1}^{n} (L_0(p_j) + L_0(p_{j-1}))(p_j - p_{j-1}) \end{aligned} \tag{7}$$

where $L_0(p_0) = 0$. Equivalence of (6) and (7) follows from

$$\begin{aligned} \frac{1}{T} \sum_{j=1}^{n} x_j (p_j + p_{j-1}) h_j &= \frac{1}{NT} \left(\sum_{j=1}^{n} x_j h_j \sum_{i=1}^{j} h_i + \sum_{j=2}^{n} x_j h_j \sum_{i=1}^{j-1} h_i \right) \\ &= \frac{1}{NT} \left(\sum_{i=1}^{n} h_i \sum_{j=i}^{n} x_j h_j + \sum_{i=1}^{n-1} h_i \sum_{j=i+1}^{n} x_j h_j \right) \end{aligned}$$

$$= \frac{1}{NT}\left(\sum_{i=1}^{n} h_i(T - TL_0(p_{i-1})) + \sum_{i=1}^{n} h_i(T - TL_0(p_i))\right)$$

$$= 2 - \sum_{j=1}^{n}(L_0(p_j) + L_0(p_{j-1}))(p_j - p_{j-1}).$$

Now suppose that the government dollar revenue requirement is C, so that $c = C/T$, and that we apply a linear income tax with marginal tax rate $r > c$. Then we have from (3) that

$$F(p_j) = rL_0(p_j) - (r - c)p_j$$

is the cumulative proportional tax through stratum j. Hence, for each stratum j, we have the following dollar amounts:

- Total Income Tax is

$$T(F(p_j) - F(p_{j-1})) = T\left(\frac{rx_jh_j}{T} - \frac{(r-c)h_j}{N}\right)$$
$$= (rx_j - (r - c)\bar{x})h_j.$$

- Average Income Tax per Household is

$$t_j = rx_j - (r - c)\bar{x}.$$

- Average Post-Tax Income per Household is

$$y_j = x_j - t_j = (1 - r)x_j + (r - c)\bar{x}.$$

- More generally, for any household with pre-tax income x , the linear dollar tax is $rx - (r - c)\bar{x}$, with percentage tax rate $r - (r - c)\bar{x}/x$. This is progressively increasing in x and approaches r for very large x.

At the other end of the scale, the tax is negative if

$$x < \left(1 - \frac{c}{r}\right)\bar{x},$$

the threshold for positive taxation. Moreover, we can use either (6) or (7) to verify (4) for the interpolated Ginis.

Using (6), we have

$$T(1 + G_0) = \sum_{j=1}^{n} x_j(p_j + p_{j-1})h_j$$

and

$$T(1 - c)(1 + G_1) = \sum_{j=1}^{n} y_j(p_j + p_{j+1})h_j$$

$$= \sum_{j=1}^{n} \left((1 - r)x_j + (r - c)\bar{x}\right)\left(p_j + p_{j-1}\right)h_j$$

$$= (1-r)T(1+G_0) + (r-c)\bar{x}\sum_{j=1}^{n}\left(p_j + p_{j-1}\right)h_j$$

$$= (1-r)T(1+G_0) + (r-c)T\sum_{j=1}^{n}\left(p_j^2 - p_{j-1}^2\right)$$

$$= (1-r)T(1+G_0) + (r-c)T$$

from which we obtain

$$G_1 = \left(\frac{1-r}{1-c}\right)G_0.$$

From the 2017 U.S. Census table [14], there are $N = 127.669$ million households and total annual household income of $T = \$11.189$ trillion, whereas average household income is $\bar{x} = \$87,643$. There are 42 strata with average household incomes ranging from \$1,128 to \$415,207. The pre-tax interpolated Gini index is $G_0 = 0.486$, close to the official Census figure of 0.482.

From the 2017 U.S. IRS table [15], we have total individual income tax revenue of $C = \$1.6053$ trillion. Combined with the Census household income data to simulate a general linear tax, we get a government tax ratio of $c = 0.14347 = 14.347\%$. With a marginal tax rate of $r = 0.50 = 50\%$, we have a common household subsidy of $(r-c)\bar{x} = \$31,248$ and a zero-tax threshold of $(1-c/r)\bar{x} = \$62,495$, below which income tax is negative. The average tax rate in the top stratum is 42% and the interpolated post-tax Gini index $G_1 = 0.284$, for an inequality reduction of 42%. So we see that significant inequality reduction is possible with a very simple linear tax structure. In Figure 4, we display pre-tax and simulated post-tax Lorenz curves based on the Census household income data, again showing significant inequality reduction in terms of twice the bounded area under the equity reference line.

The IRS table is intriguing beyond the total tax figure. There are 152.903 million individual tax returns and \$11.010 trillion total adjusted gross income (AGI), close to the Census total. There are 18 strata with average AGIs ranging from \$2,587 to \$31,259,606, a much more expansive range than the Census data. There is also a "No adjusted gross income" stratum which actually shows negative average AGI, although a relatively small number of returns pay a substantial amount of tax on average. We assume that this category is a mixture of reported capital losses on the one hand, and returns subject to the Alternative Minimum Tax on the other. For our purposes, this segment is simply ignored.

Beyond this wrinkle, the IRS table is subject to the same calculations. The interpolated pre-tax Gini index is 0.596, much higher than Census, reflecting a much more convex Lorenz curve. Moreover, the current income tax only reduces that to 0.561, a 6% reduction. In addition, the average tax rate in the top stratum is only 26%, lower than the previous few strata. Clearly, the current tax structure is not exacting on the high end and is not even progressive. We have not simulated a revised tax structure on the IRS table. In Figure 5, we display the pre-tax and post-tax Lorenz curves based on the 2017 individual tax return data. There is obviously very little inequality reduction in terms of twice the bounded area under the equity reference line.

Although the Census and IRS total pre-tax income figures are similar, the stratified tables and interpolated Lorenz curves are rather different. This difference stems from how the numbers are compiled and presented by the two bureaus. As indicated previously, the Census survey figures are widely disseminated and often quoted. Moreover, our interpolated pre-tax Gini is close to their official number. The IRS table, however,

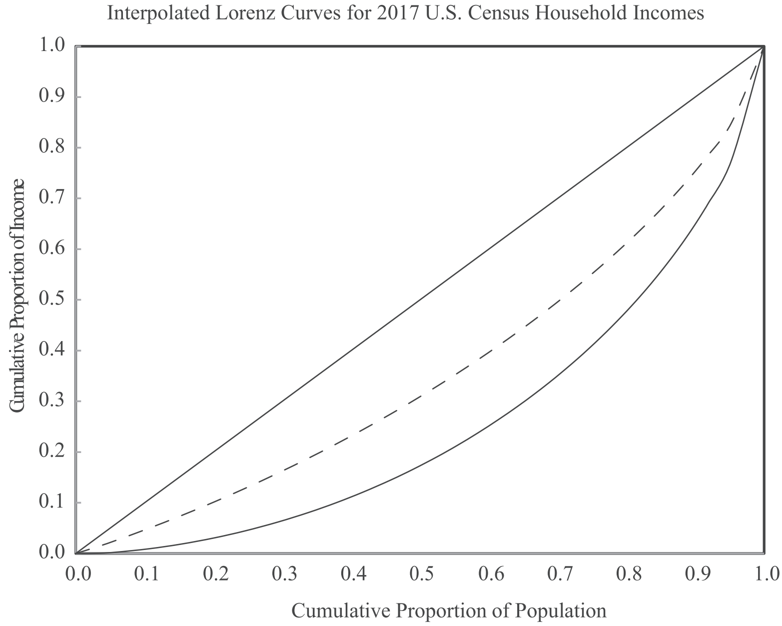

Figure 4 The solid line is the Pre-tax Lorenz curve. The dashed line is the simulated post-tax Lorenz curve.

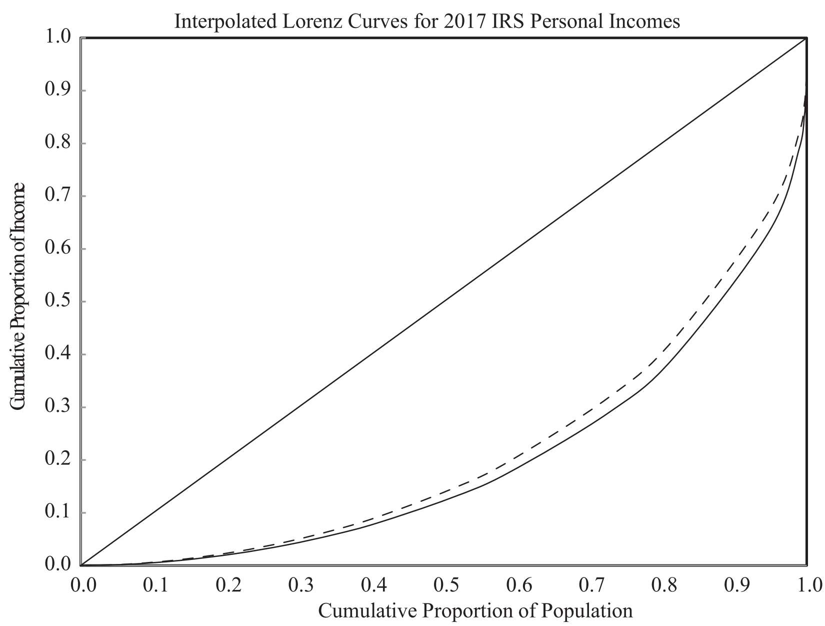

Figure 5 The solid line is the Pre-tax Lorenz curve. The dashed line is the Post-tax Lorenz curve.

is based on actual tax returns and explicitly covers a much broader range of incomes. We are not in a position to reconcile these two data sources, but we have found each to be individually useful in our research and they are corroborative in a general sense. We would like to thank both of these agencies for providing their data for analysis.

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[15] U. S. Internal Revenue Service. (2017). Table 1.1: All Returns: Selected Income and Tax Items by Size and Accumulated Size of Adjusted Gross Income, Tax Year 2017. www.irs.gov/statistics/soi-tax-stats-individual-statistical-tables-by-size-of-adjusted-gross-income. Download Excel file is 17in11si.xls.

Summary. Income inequality measurement via the Gini index is reviewed, both pre-tax and post-tax for a general tax curve. Focus then shifts to the linear income tax with a single marginal tax rate and a common subsidy for all taxpayers. It is demonstrated that significant inequality reduction is possible with this simple tax structure. Finally, we benchmark against real incomes data from the U.S. Census Bureau and the U.S. Internal Revenue Service.

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Cycloids Generated by Rolling Regular Polygons

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Imagine rolling a wheel with a point marked on it. Visualize the marked point as continuously leaving a trace of its location. The shape of the curve that is drawn depends upon (1) the shape of the wheel, (2) the placement of the tracing point on the wheel, and (3) the curvature of the path traversed by the wheel. For example, a point placed on the perimeter of a circular wheel when rolled along a straight line path traces the well known *cycloid*, as shown in Figure 1.

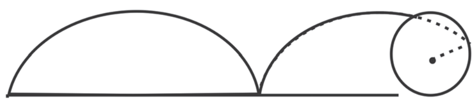

Figure 1 The marked point on the rolling wheel traces out a cycloid.

While it is not known when, in antiquity, the cycloid was first studied, Christiaan Huygens proved in 1673 that the cycloid solved the famous tautochrone problem. Newton, Leibnitz, and others demonstrated in 1696 that the cycloid also solved the equally famous brachistochrone problem. Much of this history is recounted in the excellent article by Martin [**2**].

Instead of rolling circular wheels along straight lines or circular paths, an interesting variation arises from "rolling" regular n-sided polygons along straight lines. Placing a tracing point at the midpoint of one polygonal side results in a traced path composed of joined circular arcs that form the arches of a curve that we shall call a *polycycloid*, as shown in Figure 2.

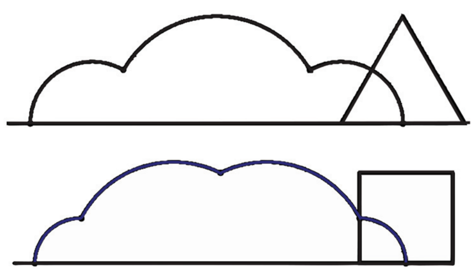

Figure 2 Examples of polycycloids.

In a noteworthy 1992 article by Leon Hall and Stan Wagon in this MAGAZINE, the authors investigated the shape required of a road to allow a polygonal wheel to

Math. Mag. **94** (2021) 267–272. doi:10.1080/0025570X.2021.1951536 © Mathematical Association of America
MSC: 26A06

roll smoothly [1]. Subsequently, in a widely read 1999 *Math Horizons* piece, Wagon showed that a bicycle with square wheels would roll smoothly along a road constructed as a sequence of inverted catenary curves [3].

Whereas the focus of these prior articles was on the required road shapes for a wheel to roll smoothly, this present piece investigates not the road, but (1) the length of traced polycycloidal paths, and (2) the area bounded by a polycycloid above and by the x-axis below. The analysis involves limit techniques central to the calculus and it inspires a number of related challenge problems for interested students, as presented at the end of this article.

Arc Length

For a cycloid created by rolling a circle of radius $r = 1$, with tracing point located on the perimeter of the circle, the horizontal displacement D of the tracing point defined by one revolution of the circle equals the circles circumference. Specifically, $D = 2\pi(r) = 2\pi(1) = 2\pi$

Using the parametric equations for a cycloid, $x = r(\theta - \sin\theta)$ and $y = r(1 - \cos\theta)$, the following integral shows that the length L of the arc generated by rolling the circle through one revolution is $8r$:

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} r\sqrt{2 - 2\cos\theta}\, d\theta$$
$$= \int_0^{2\pi} 2r \sin\frac{\theta}{2}\, d\theta = 8r.$$

When $r = 1$, the arc length L for one arch of the cycloid formed by rolling a circle is $L = 8$.

In order to compute the length L_n of one arch of a polycycloid, we need the radius of each of the component circular arcs. For convenience, place a regular n-gon and a circumscribing circle of radius r so that the circle and n-gon are both centered at the origin (0, 0), as shown in Figure 3.

The rotation angle θ defined by rolling the n-gon about a vertex as the n-gon rolls from one side onto its adjacent side is $\theta = \frac{2\pi}{n}$

The radius r of the circumscribing circle must be determined from the size of the n-gon with side length s_n, prescribed by the requirement that the perimeter of the polygon must equal the circumference of a unit circle if the resulting polycycloids are to start and end at the same locations.

The distance formula provides expressions for lengths r_n and s_n identified in Figure 3.

$$r_n = r\sqrt{\left(\frac{\cos(\frac{2k\pi}{n}) + \cos(\frac{2(k-1)\pi}{n}) - 2)}{2}\right)^2 + \left(\frac{\sin(\frac{2k\pi}{n}) + \sin(\frac{2(k-1)\pi}{n})}{2}\right)^2}$$

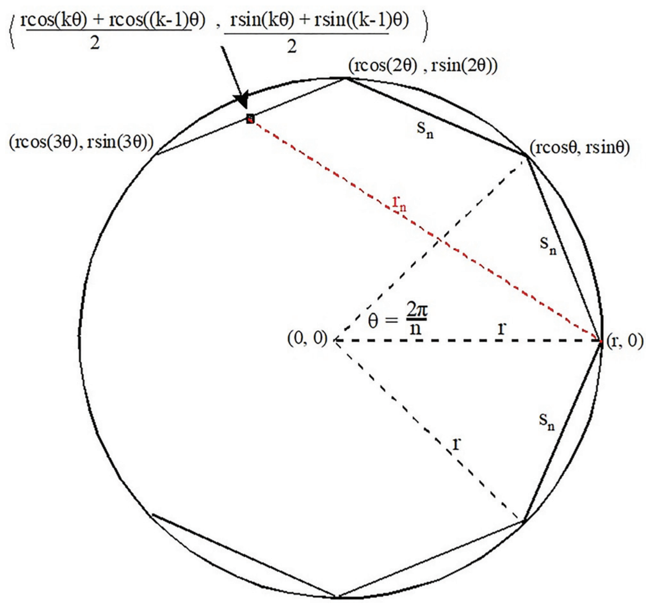

Figure 3 A regular n-gon with a circumscribed circle, centered at the origin.

$$s_n = \sqrt{(r\cos\theta - r)^2 + (r\sin\theta)^2} = \sqrt{r^2\cos^2\theta - 2r^2\cos\theta + r^2 + r^2\sin^2\theta}$$

$$= r\sqrt{\cos^2\theta - 2\cos\theta + 1 + \sin^2\theta} = r\sqrt{2 - 2\cos\theta}$$

$$= r\sqrt{4\sin^2\left(\frac{\theta}{2}\right)} = 2r\sin\left(\frac{\theta}{2}\right).$$

Letting $\theta = 2\pi/n$, the relationship between s_n and r simplifies to $s_n = 2r\sin(\pi/n)$, and therefore, $r = s_n/(2\sin(\pi/n))$.

Since $s_n = D/n$, it follows that $r = D/(2n\sin(\pi/n))$, and the total polycycloid arc length L_n is given by

$$L_n = \sum_{k=1}^{n}\theta r_k = \sum_{k=1}^{n}\frac{2\pi r_k}{n} = \frac{2\pi}{n}\sum_{k=1}^{n} r_k = \left(\frac{2\pi}{n}\right)\left(\frac{D}{2n\sin(\frac{\pi}{n})}\right)\times$$

$$\sum_{k=1}^{n}\sqrt{\left(\frac{\cos(\frac{2k\pi}{n}) + \cos(\frac{2(k-1)\pi}{n}) - 2)}{2}\right)^2 + \left(\frac{\sin(\frac{2k\pi}{n}) + \sin(\frac{2(k-1)\pi}{n})}{2}\right)^2}$$

$$= \left(\frac{1}{2}\right)\left(\frac{2\pi}{n}\right)\left(\frac{2\pi}{2n\sin(\frac{\pi}{n})}\right)\sum_{k=1}^{n}\sqrt{A+B},$$

where

$$A = \left(\cos\left(\frac{2k\pi}{n}\right) + \cos\left(\frac{2(k-1)\pi}{n}\right) - 2\right)^2$$

$$B = \left(\sin\left(\frac{2k\pi}{n}\right) + \sin\left(\frac{2(k-1)\pi}{n}\right)\right)^2.$$

Since regular n-gons become increasingly circular as n approaches infinity, the length L_n of one arch of a polycycloid must converge to $L = 8$, which is the length of one arch of a cycloid generated by rolling a circle of radius $r = 1$. See Figure 4.

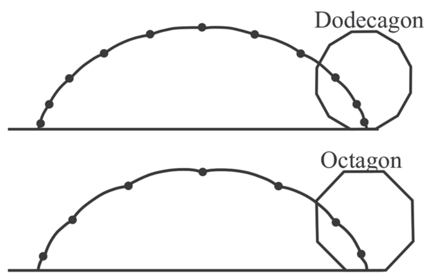

Figure 4 Polycycloids with $n = 8$ and $n = 12$.

Using the L_n formula, values of L_n for regular n-gon polycycloids can be seen to converge to $L = 8$ as n increases, just as expected. This is shown in Table 1.

TABLE 1: Arc length L_n of one arch of a polycycloid for different values of n.

n sides	L_N
3	8.185303
4	7.984678
5	7.952617
6	7.951943
7	7.957557
8	7.963763
12	7.980583
30	7.996458
100	7.999672
1000	7.999997

Despite the apparent pattern revealed in Table 1, many mathematics students would find it challenging to prove that $\lim_{n\to\infty} L_n = 8$ even with the powerful methods of calculus at their disposal.

Area

As the number of sides n of a polygon increases, the area A_n under one arch of a polycycloid traced by the rolling n-sided regular polygon approaches the area A under a cycloid traced by rolling a circle. In order to compute the area A under one arch of a cycloid traced by rolling a circle of radius $r = 1$, we again use the parametric equations of the cycloid, $x = r(\theta - \sin\theta)$ and $y = r(1 - \cos\theta)$ with $dx = r(1 - \cos\theta)\, d\theta$, to write an integral that computes the required area:

$$A = \int_{\theta=0}^{\theta=2\pi} y\, dx = \int_0^{2\pi} r^2(1 - \cos\theta)^2\, d\theta = 3\pi r^2$$

For a circle of radius $r = 1$, the area A under the cycloid reduces to $A = 3\pi(1)^2 = 3\pi \approx 9.424778$.

Now we compute the area A_n under a polycycloid generated by a regular n-gon. In contrast to the cycloid formed by rolling a circle, a pattern of successive circular arcs join to form the polycycloidal curve generated by the n-gon. In Figure 5, each arc is translated to the n-gon's original starting position.

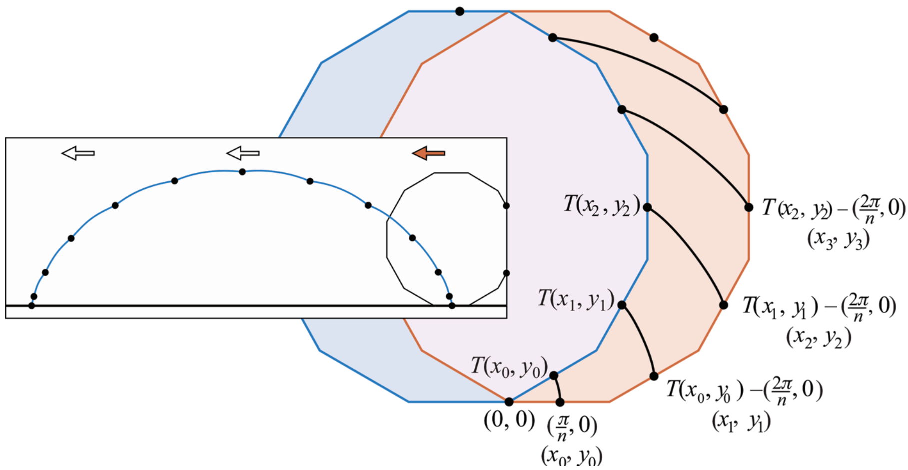

Figure 5 Endpoint coordinates associated with successive arcs of a polycycloid generated by an n-gon.

We find the successive integral limits by rotating the n-gon through counterclockwise rotations of $2\pi/n$ radians starting with $(x_0, y_0) = (\pi/n, 0)$. For $1 \leq k \leq n$, we define a point $T(x_k, y_k)$ according to the recursive formulas:

$$x_{k+1} = x_k \cos\frac{2\pi}{n} - y_k \sin\frac{2\pi}{n} \qquad \text{and} \qquad y_{k+1} = x_k \sin\frac{2\pi}{n} + y_k \cos\frac{2\pi}{n}.$$

We then apply a horizontal translation of $2\pi/n$ to the point $T(x_k, y_k)$ as shown in Figure 5.

The area A_n under one complete cycle of a polycycloid can be determined by summing the areas bounded above by each of the circular arcs (that together form the polycycloid) and below by the horizontal axis.

$$A_n = 2\left(\int_{T(x_0,y_0)}^{x_0} \sqrt{r_1^2 - x^2}\,dx + \sum_{k=1}^{n/2} \int_{T(x_k,y_k)}^{T(x_{k-1},y_{k-1})+(\frac{2\pi}{n},0)} \sqrt{r_{k+1}^2 - x^2}\,dx\right),$$

where the integral limits represent the x-coordinates of the aforementioned points. (See Figure 5).

Again, as the number of sides of a regular n-gon increases, the shape of the n-gon converges to that of a circle. Accordingly, the area A_n generated by rolling a regular polygon must approach that of the cycloid traced by rolling a circle.

Using the area formula for A_n with successively larger values of n, the polycloid area A_n increases as expected to $A = 9.424778$, the area defined by the cycloid (obtained by rolling a circle). The areas for a few values of n are shown in Table 2.

TABLE 2: The area A_n for a polycycloid generated by a polygon with n sides.

n Sides	A_n
3	7.641309
4	8.280178
8	9.1103217
12	9.2828896
Circle	9.424778

Possibilities for further investigation

Students wishing to further explore aspects of polycycloids might choose to attempt some of the following problems:

1. As $n \to \infty$, the number of cusps that occur on a polycycloid increases (that is, the number of points where the path is non-differentiable increases) suggesting that the final curve is everywhere non-differentiable. On the other hand, as $n \to \infty$, polycycloids converge to the smooth path of the everywhere differentiable cycloid. Does this suggest that, as $n \to \infty$, the polycycloids yield a curve that is both everywhere differentiable and simultaneously everywhere non-differentiable?
2. Investigate curves produced by rolling regular n-gons around circular path. For comparison, curves formed by rolling a circle around the outside of another circle are called epicycloids. If a circle is rolled around the inside of a containing circle, then the resulting curve is called a hypocycloid.
3. Evaluate $\lim_{n\to\infty} L_n$ and $\lim_{n\to\infty} A_n$.

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Summary. Historical problems related to cycloids form the background for an investigation of paths traced by rolling regular polygons. Using trigonometry, geometry, and calculus, the lengths of and areas under the generated paths are shown to converge to values associated with the classical cycloid as the number of polygonal sides increases.

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A Nearly Forgotten Formula: Halphen's Identity for Derivatives and Beyond

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We bring to light a nearly forgotten identity which was found by Georges-Henri Halphen in 1880 [4]. In order to facilitate his investigations of the nth derivative of $x^m e^{1/x}$, he developed the amazing identity

$$\left(f\left(\frac{1}{x}\right)g\left(x\right)\right)^{(n)}=\sum_{k=0}^{n}\left(-1\right)^k\binom{n}{k}\frac{1}{x^k}f^{(k)}\left(\frac{1}{x}\right)\left(\frac{g\left(x\right)}{x^k}\right)^{(n-k)} \tag{1}$$

with $n=0,1,2,\ldots$, for sufficiently differentiable functions. Equation (1) appears as an exercise in Comtet's book [1, Exercise 15, p. 161] and with a proof in the recent book by Quaintance and Gould [6, Theorem 8.2]. Various choices of the involved functions deliver interesting formulas. It will be pointed out that Halphen's identity (1) generalizes several special results occurring in the literature.

By taking the nth derivatives of the functions $e^{\sqrt{x}}$, e^{x^2} or $e^{1/x}$, one gets expressions consisting of the same functions multiplied by a sum of various powers of x. A closer look reveals that this sum is a polynomial in the variables $\sqrt{x}$, x^2, or $1/x$, respectively, multiplied by x^{-n}. More generally, if α is any real constant, the first derivatives of e^{x^α} are given by

$$\left(\frac{d}{dx}\right)^1 e^{x^\alpha}=\alpha x^{\alpha-1}e^{x^\alpha},$$

$$\left(\frac{d}{dx}\right)^2 e^{x^\alpha}=\left(\alpha\left(\alpha-1\right)x^{\alpha-2}+\alpha^2x^{2\alpha-2}\right)\cdot e^{x^\alpha},$$

and continuing this procedure leads to the insight that

$$\left(\frac{d}{dx}\right)^n e^{x^\alpha}=x^{-n}P_n\left(x^\alpha\right)\cdot e^{x^\alpha}$$

for a certain polynomial P_n of degree less than or equal to n which is independent of α. In order to obtain an explicit representation the question arises: What exactly are the coefficients of P_n?

Moving from the exponential function to an arbitrary smooth function f, one could ask for an explicit expression of the nth derivative of $f\left(x^\alpha\right)$. This question will be studied in the third section.

In a recent article, Daboul, Mangaldan, Spivey, and Taylor studied the nth derivative of $e^{1/x}$ [2]. Using direct calculation for small values of n and looking for a pattern, they found the formula

$$\frac{d^n}{dx^n}\left(e^{1/x}\right)=\left(-1\right)^n e^{1/x}\sum_{k=1}^{n}\binom{n}{k}\binom{n-1}{k-1}\frac{(n-k)!}{x^{n+k}}\quad(n=1,2,\ldots). \tag{2}$$

Math. Mag. **94** (2021) 273–280. doi:10.1080/0025570X.2021.1951537 © Mathematical Association of America
MSC: 26A24 (Differentiation (functions of one variable))

This can be rewritten in the concise form

$$\frac{d^n}{dx^n}\left(e^{1/x}\right) = e^{1/x}\sum_{k=1}^{n}\frac{L_{n,k}}{x^{n+k}} \qquad (n = 1, 2, \ldots), \tag{3}$$

where $L_{n,k}$ denote the Lah numbers given by

$$L_{n,k} = (-1)^n \frac{n!}{k!}\binom{n-1}{k-1}.$$

The authors gave five proofs of this formula, using different properties of the Lah numbers. In the process, they took an interesting tour through several areas of mathematics including Faà di Bruno's formula, set partitions, Maclaurin series, factorial powers, the Poisson probability distribution, and hypergeometric functions.

Obviously, equation (2) is a direct consequence of Halphen's identity (1). In the special case $f(x) = e^x$, equation (1) reduces to

$$\frac{d^n}{dx^n}\left(g(x)\cdot e^{1/x}\right) = e^{1/x}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{x^k}\left(\frac{g(x)}{x^k}\right)^{(n-k)}. \tag{4}$$

Putting $g(x) = 1$ and observing that

$$\left(x^{-k}\right)^{(n-k)} = (-1)^{n-k}\frac{(n-1)!}{(k-1)!}x^{-n}$$

for $k = 1, 2, \ldots, n$, immediately yields equation (2).

An interesting example already given by Halphen [4, Eq. (3)] is the formula

$$\left(\frac{d}{dx}\right)^n\left(x^{n-1}e^{1/x}\right) = (-1)^n\frac{1}{x^{n+1}}e^{1/x}.$$

Halphen proved his identity (1) by verifying it for monomials. The general case is then based on a density argument. In the next section, we reproduce the original proof [4] in a slightly improved form. More generally, in the fourth section we deduce a formula for the derivatives of $f(x^\alpha)\, g\left(x^\beta\right)$, where α, β are any real constants. As a consequence of this formula, we obtain a direct proof of equation (1) which establishes the identity for arbitrary functions of sufficient smoothness. Our derivation is constructive in the sense that it provides not only a verification of a known result, but it leads to Halphen's formula in a natural way. This will be presented in the last section.

The original proof of Halphen's identity

In his original proof of equation (1), Halphen supposed that both functions f and g are analytic. Then he proved his formula by verifying that it is valid when f and g are monomials, and hence also for polynomials. From this, the identity extends to non-analytic functions of sufficient smoothness since equation (1) is of an algebraic nature among the derivatives, and because the polynomials are dense in the space of continuous functions. The celebrated Weierstrass approximation theorem states that each continuous function on a closed interval I can be uniformly approximated by algebraic polynomials with arbitrary precision. By approximating $f^{(n)}$ on I, we can choose a polynomial p which simultaneously approximates $f \in C^n(I)$. That is, for given $\varepsilon > 0$, we have

$$\left|f^{(k)}(x) - p^{(k)}(x)\right| < \varepsilon \qquad (x \in I,\ k = 0, \ldots, n).$$

Note that Weierstrass published his theorem in 1885 [7], five years after Halphen's paper.

In this section, we present Halphen's proof in a slightly simplified form. For $x \neq 0$, put

$$f\left(\frac{1}{x}\right) = \sum_{i=0}^{\infty} a_i x^{-i} \qquad \text{and} \qquad g(x) = \sum_{j=0}^{\infty} b_j x^j.$$

Direct calculation shows that the left-hand side of equation (1) satisfies

$$\left(f\left(\frac{1}{x}\right) g(x)\right)^{(n)} = n! \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_i b_j \binom{j-i}{n} x^{j-i-n}. \tag{5}$$

The right-hand side of equation (1) is equal to

$$n! \sum_{k=0}^{n} (-1)^k \frac{1}{x^k} \sum_{i=0}^{\infty} a_i \binom{i}{k} \left(\frac{1}{x}\right)^{i-k} \sum_{j=0}^{\infty} b_j x^{j-k-(n-k)} \binom{j-k}{n-k}$$
$$= n! \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_i b_j x^{j-i-n} \sum_{k=0}^{n} (-1)^k \binom{i}{k} \binom{j-k}{n-k}.$$

Taking advantage of the inflection formula

$$\binom{z}{n} = (-1)^n \binom{n-z-1}{n}$$

for binomial coefficients and the Vandermonde convolution

$$\sum_{k=0}^{n} \binom{a}{k} \binom{b}{n-k} = \binom{a+b}{n},$$

we can simplify the inner sum:

$$\sum_{k=0}^{n} (-1)^k \binom{i}{k} \binom{j-k}{n-k} = (-1)^n \sum_{k=0}^{n} \binom{i}{k} \binom{n-j-1}{n-k}$$
$$= (-1)^n \binom{n+i-j-1}{n} = \binom{j-i}{n}.$$

Comparison with equation (5) completes the proof.

The proof given by Quaintance and Gould [6, pages 110–112] uses neither the density of polynomials nor the analyticity of the functions. Instead, they apply Hoppe's representation of Faà di Bruno's formula for the derivatives of composite functions. Unfortunately, their proof is too long to reproduce here. We shall give another proof of Halphen's identity by deducing it, for arbitrary sufficiently smooth functions, from a more general result.

Derivatives of $f(x^\alpha)$

Having studied the derivatives of the function $f(1/x)$, we now consider, more generally, functions of the type $f(x^\alpha)$, for arbitrary constants α. We recall Faà di Bruno's formula for the derivatives of composite functions. If f and g are functions with a sufficient number of derivatives, then

$$\left(\frac{d}{dx}\right)^n f(g(x)) = \sum_{m=0}^{n} f^{(m)}(g(x))\, n! \sum \prod_{\nu=1}^{n} \frac{1}{c_\nu!}\left(\frac{g^{(\nu)}(x)}{\nu!}\right)^{c_\nu},$$

where $n \in \mathbb{N}$, and where the inner sum is over all different solutions in nonnegative integers $c_1, \ldots, c_n$ of

$$c_1 + \cdots + c_n = m \qquad \text{and} \qquad c_1 + 2c_2 + \cdots + nc_n = n$$

(see, for example, Comtet [1, p. 137]). In the special case $g(x) = x^\alpha$ we immediately obtain

$$\left(\frac{d}{dx}\right)^n f(x^\alpha) = \sum_{m=0}^{n} f^{(m)}(x^\alpha)\, x^{m\alpha-n} Z_{n,m}(\alpha), \tag{6}$$

for certain real numbers $Z_{n,m}(\alpha)$. Of course, equation (6) can easily be shown by mathematical induction on n. This procedure yields the linear recurrence

$$Z_{n+1,m}(\alpha) = \alpha Z_{n,m-1}(\alpha) + (m\alpha - n)\, Z_{n,m}(\alpha)$$

with the convention that $Z_{n,m}(\alpha) = 0$ if $m > n$ or $n < 0$. Equation (6) appears in Comtet's book [1, Ex. 7, p. 157f], where the $Z_{n,m}(\alpha)$ are given via an exponential generating function. Without proof, Comtet states that

$$\sum_{n=m}^{\infty} Z_{n,m}(\alpha)\frac{z^n}{n!} = \frac{1}{m!}\left((1+z)^\alpha - 1\right)^m. \tag{7}$$

Since we do not know a reference for a proof, we deduce this formula in the following exposition.

Let $x \neq 0$. Again supposing that f is analytic, series expansion yields

$$f((x+z)^\alpha) = \sum_{n=0}^{\infty}\left[\left(\frac{d}{dx}\right)^n f(x^\alpha)\right]\frac{z^n}{n!}, \tag{8}$$

for sufficiently small z. On the other hand we have, for $|z| < |x|$,

$$\begin{aligned} f((x+z)^\alpha) &= f\left(x^\alpha + x^\alpha\left(1+\frac{z}{x}\right)^\alpha - x^\alpha\right) \\ &= \sum_{m=0}^{\infty}\frac{f^{(m)}(x^\alpha)}{m!}\left(x^\alpha\left(1+\frac{z}{x}\right)^\alpha - x^\alpha\right)^m. \end{aligned}$$

Using the numbers defined by equation (7) we obtain

$$\begin{aligned} f((x+z)^\alpha) &= \sum_{m=0}^{\infty} f^{(m)}(x^\alpha)\, x^{m\alpha}\frac{1}{m!}\left(\left(1+\frac{z}{x}\right)^\alpha - 1\right)^m \\ &= \sum_{m=0}^{\infty} f^{(m)}(x^\alpha)\, x^{m\alpha}\sum_{n=m}^{\infty} Z_{n,m}(\alpha)\frac{1}{n!}\left(\frac{z}{x}\right)^n \\ &= \sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{m=0}^{n} f^{(m)}(x^\alpha)\, x^{m\alpha-n} Z_{n,m}(\alpha). \end{aligned}$$

Comparison with equation (8) shows that formula (6) is valid with $Z_{n,m}(\alpha)$, as defined in equation (7).

We leave it as an exercise to the reader to calculate that

$$Z_{n,m}(1/2) = (-1)^{n-m} \frac{(n-1)!}{(m-1)!} \binom{2n-m-1}{n-1} \frac{1}{2^{2n-m}},$$

$$Z_{n,m}(2) = \frac{n!}{m!} \binom{m}{n-m} 2^{2m-n},$$

$$Z_{n,m}(-1) = (-1)^n \frac{n!}{m!} \binom{n-1}{m-1} = L_{n,m}.$$

Note that $Z_{n,m}(-1)$ coincides with the Lah number $L_{n,m}$. By equation (6) we obtain, for $f(x) = e^x$ and $\alpha \in \{1/2, 2, -1\}$,

$$\left(\frac{d}{dx}\right)^n e^{\sqrt{x}} = e^{\sqrt{x}} \sum_{m=1}^{n} (-1)^{n-m} \frac{(n-1)!}{(m-1)!} \binom{2n-m-1}{n-1} \frac{1}{(4x)^{n-m/2}} \quad (n \geq 1),$$

$$\left(\frac{d}{dx}\right)^n e^{x^2} = e^{x^2} \sum_{m=0}^{n} \frac{n!}{m!} \binom{m}{n-m} (2x)^{2m-n} \quad (n \geq 0),$$

$$\left(\frac{d}{dx}\right)^n e^{1/x} = (-1)^n e^{1/x} \sum_{m=0}^{n} \frac{n!}{m!} \binom{n-1}{m-1} x^{-m-n} \quad (n \geq 1).$$

The right-hand side of the second formula is equal to $(-i)^n H_n(ix) e^{x^2}$, where H_n denotes the nth Hermite polynomial and i is the imaginary unit. The latter formula is just equation (2).

Derivatives of $f(x^\alpha) g(x^\beta)$

We now derive a formula for the nth derivative of $f(x^\alpha) g(x^\beta)$. Application of the Leibniz rule and equation (6) leads to

$$\begin{aligned}
&\left(f(x^\alpha) g(x^\beta)\right)^{(n)} \\
&= \sum_{k=0}^{n} \binom{n}{k} \left(\sum_{i=0}^{k} f^{(i)}(x^\alpha) x^{i\alpha-k} Z_{k,i}(\alpha)\right) \left(\sum_{j=0}^{n-k} g^{(j)}(x^\beta) x^{j\beta-(n-k)} Z_{n-k,j}(\beta)\right) \\
&= \sum_{i=0}^{n} \sum_{j=0}^{n-i} f^{(i)}(x^\alpha) g^{(j)}(x^\beta) x^{i\alpha+j\beta-n} \sum_{k=0}^{n} \binom{n}{k} Z_{k,i}(\alpha) Z_{n-k,j}(\beta).
\end{aligned}$$

In the following, we use the notation:

$$[z^n] h(z) = [z^n] \sum_{\nu=0}^{\infty} h_\nu z^\nu = h_n$$

for the coefficient of z^n in a formal power series

$$h(z) = \sum_{\nu=0}^{\infty} h_\nu z^\nu.$$

By equation (7), we obtain

$$\sum_{k=0}^{n}\binom{n}{k}Z_{k,i}(\alpha)Z_{n-k,j}(\beta)$$
$$=\frac{n!}{i!j!}\sum_{k=0}^{n}\left(\left[z^{k}\right]((1+z)^{\alpha}-1)^{i}\right)\left(\left[z^{n-k}\right]\left((1+z)^{\beta}-1\right)^{j}\right)$$
$$=\frac{n!}{i!j!}\left[z^{n}\right]\left(((1+z)^{\alpha}-1)^{i}\left((1+z)^{\beta}-1\right)^{j}\right),$$

where in the last step we applied the Cauchy product. Finally, we obtain

$$\left(f(x^{\alpha})\,g\left(x^{\beta}\right)\right)^{(n)}=n!\sum_{i=0}^{n}\sum_{j=0}^{n-i}\frac{f^{(i)}(x^{\alpha})}{i!}\frac{g^{(j)}\left(x^{\beta}\right)}{j!}x^{i\alpha+j\beta-n}\times \left[z^{n}\right]\left(((1+z)^{\alpha}-1)^{i}\left((1+z)^{\beta}-1\right)^{j}\right). \tag{9}$$

The step to a product of many functions, that is, the formula

$$\left(\frac{d}{dx}\right)^{n}\left(\prod_{k=1}^{r}f_{k}(x^{\alpha_{k}})\right)= \frac{n!}{x^{n}}\sum_{i_{1}+\cdots+i_{r}=n}\left(\prod_{k=1}^{r}\frac{f_{k}^{(i_{k})}}{i_{k}!}(x^{\alpha_{k}})\,x^{i_{k}\alpha_{k}}\right)\left[z^{n}\right]\prod_{k=1}^{r}((1+z)^{\alpha_{k}}-1)^{i_{k}},$$

where $\alpha_1, \ldots, \alpha_r$ are real constants, is immediate.

A direct proof of Halphen's identity

As we have seen, Halphen's original proof verifies equation (1) only for monomials, while its full generality follows by a density argument. We present a short direct proof that equation (1) is valid for arbitrary functions of sufficient smoothness.

Let us consider the special case $\alpha = -1$, $\beta = 1$ in equation (9). To this end, we derive, for these values of α and β, an explicit expression of the series coefficient

$$\left[z^{n}\right]\left(((1+z)^{\alpha}-1)^{i}\left((1+z)^{\beta}-1\right)^{j}\right)$$

in terms of a certain binomial coefficient. We have

$$\left[z^{n}\right]\left(\left((1+z)^{-1}-1\right)^{i}\left((1+z)^{1}-1\right)^{j}\right)$$
$$=\left[z^{n}\right]\left(\left(\frac{-z}{1+z}\right)^{i}z^{j}\right)=(-1)^{i}\left[z^{n}\right]\frac{z^{i+j}}{(1+z)^{i}}.$$

For $i \geq 1$, the well-known series expansion

$$(1+z)^{-i}=\sum_{\nu=0}^{\infty}\binom{\nu+i-1}{\nu}(-z)^{\nu}$$

leads to

$$[z^n]\left(\left((1+z)^{-1}-1\right)^i\left((1+z)^1-1\right)^j\right)$$
$$=\begin{cases}0 & \text{if } i+j>n\\ (-1)^{n-j}\binom{n-j-1}{n-j-i} & \text{if } 0\le i+j\le n.\end{cases}$$

We remark that this equation is also valid in the case $i=0$, since $\binom{n-j-1}{n-j-i}=0$ for $0\le j\le n-1$, and $\binom{n-j-1}{n-j-i}=1$, for $j=n$. Hence, equation (9) takes the form

$$\left(f\left(\frac{1}{x}\right)g(x)\right)^{(n)}=$$
$$n!\sum_{i=0}^{n}\frac{f^{(i)}\left(\frac{1}{x}\right)}{i!}\sum_{j=0}^{n-i}\frac{g^{(j)}(x)}{j!}x^{-i+j-n}(-1)^{n-j}\binom{n-j-1}{n-j-i}.$$

Noting that

$$\frac{1}{(n-j-i)!}\left(x^{-i}\right)^{(n-i-j)}=\binom{-i}{n-j-i}x^{-n+j}$$
$$=(-1)^{n-i-j}\binom{n-j-1}{n-j-i}x^{-n+j}$$

(the last step used the aforementioned inflection formula for binomial coefficients), we conclude that the inner sum is equal to

$$\frac{(-1)^i x^{-i}}{(n-i)!}\sum_{j=0}^{n-i}\binom{n-i}{n-i-j}g^{(j)}(x)\left(x^{-i}\right)^{(n-i-j)}$$
$$=\frac{(-1)^i x^{-i}}{(n-i)!}\left(\frac{g(x)}{x^i}\right)^{(n-i)},$$

where the latter equation follows by an application of the Leibniz rule. Finally, we obtain

$$\left(f\left(\frac{1}{x}\right)g(x)\right)^{(n)}=\sum_{i=0}^{n}(-1)^i\binom{n}{i}\frac{f^{(i)}\left(\frac{1}{x}\right)}{x^i}\left(\frac{g(x)}{x^i}\right)^{(n-i)}$$

which is Halphen's identity (1).

Acknowledgments The author is grateful to the anonymous reviewer and the editor. Their valuable advice enriched the paper by suggesting additional explanations, leading to a better exposition.

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Summary. We bring to light a nearly forgotten identity found by Halphen in 1880. A special case is an expression for the nth derivative of $\exp(1/x)$. This formula was studied by Daboul, Mangaldan, Spivey, and Taylor in their recent paper in this MAGAZINE. We reproduce Halphen's original proof in a slightly improved form, which verifies his identity for monomial functions. More generally, we study derivatives of $f(x^\alpha)$ and establish a formula for the higher order derivatives of the product $f(x^\alpha)g(x^\beta)$. As a consequence, we obtain a new proof of Halphen's result which establishes the identity for arbitrary functions of sufficient smoothness.

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Triangular Numbers and the N-Sided Die Optimal Stopping Problem

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A fair die with sides numbered from one to one hundred is rolled. You can either receive the value shown or else pay one and roll again. This can be repeated as many times as you wish. The questions are, "When should you roll again?" and "What is the maximum expected return or gain from playing this game?" This is the 100-sided die optimal stopping game. It has become a popular problem on the internet with several solutions given in [5], [6].

The problem is an example of one of the earliest optimal stopping problems and can be traced back to the 1950s (see, for example, the paper by Sakaguchi [4]). In those problems, the rolls of the die are replaced with values of independent, identically distributed random variables with interpretations ranging from bids for houses to salary offers. Rolling again corresponds to rejecting an offer, and the cost is the cost of waiting. Techniques for solving the general problem have been given (for example, Chow and Robinson [1], Ferguson [2], and Ferguson and Klass [3]).

What is left to be done? The answer is that the solutions for the 100-sided die, and more generally the N-sided die, optimal stopping problems lead to interesting questions specific to rolling a die that are not present in the general problem. In particular, the relation between the value of the game and the optimal stopping rule is more complicated than in the continuous case. Because of this, the solutions on the internet for the 100-sided die problem use numerical approximations to the solution. This is not necessary and obscures properties of the general solution.

In this paper, we give exact formulas for the optimal stopping rule and maximal expected gain or value V of the game. We also show that there is an elegant, simple pattern for these solutions as a function of N that is related to triangular numbers. A *triangular number* is a number of the form $N = 0 + 1 + 2 + \cdots + m$, in which case m is called the *triangular root* of N. In particular, we find that V is an integer if and only if N is a triangular number. In this case alone, the optimal stopping rule in the sense above is not unique. Nevertheless, we find a good way to make a choice among these competing rules.

In the N-sided die problem, there is a dual problem whose solution is related to that of the original problem. In this case, you roll the die and pay the amount shown rather than receive the amount shown. As before, you may pay one and roll again. The object is to minimize the expected total payment. We call this the *minimal expected loss problem* in contrast to the original problem, which we call the the *maximal expected gain problem*. The elegant pattern of the solutions for the minimal expected loss problem is much more apparent than the pattern for the maximal expected gain problem, so we start there. The solutions for the gain problem then follow easily. For the gambler concerned with more than the expected gains or losses, we also find the limiting distribution for normalized losses and gains as N approaches infinity.

Math. Mag. **94** (2021) 281–287. doi:10.1080/0025570X.2021.1951574
This article has been corrected with minor changes. These changes do not impact the academic content of the article.
MSC: 60G40

The minimal expected loss problem.

We begin with the more general problem of rolling an N-sided die, where we must pay a positive integer c for each additional roll, and the goal is to minimize the expected loss. Let L be the loss, and $V = E(L)$ be the expected loss, using the optimal strategy or value of the game. If we stop after the first toss X_1, then the loss is X_1. If we continue, then the expected loss is $c + V$, since we pay c to play again and the game begins anew. It then follows that we should stop after the first toss if $X_1 < c + V$, and continue if $X_1 > c + V$. One can do either if $X_1 = c + V$. This can only happen if V is an integer. In this case, the optimal stopping rule is not unique.

For the purpose of computing the value V of the game, we assume that we stop if $X_1 \leq \lfloor V + c \rfloor$, the greatest integer less than or equal to $V + c$, and continue otherwise. If V is an integer, one could also continue if $X_1 = c + V$, but that rule would lead to a game with the same value V.

If $c > N$, then we would always stop after the first roll, so we assume $c \leq N$. For many of the results of this paper, we will use the unique triangular representation of cN in terms of the largest triangular number less than or equal to cN. This give us $cN = 1 + \cdots + (c + k - 1) + j$ for k a positive integer and j an integer from 0 to $c + k - 1$. This follows because $1 + \cdots + c \leq cN$. The first theorem uses this representation to give a way to compute the exact value of these games without having to use numerical approximations, as is done in the internet solutions for $N = 100$ and $c = 1$.

Theorem 1. *Let $cN = 1 + \cdots + (c + k - 1) + j$, for $k \geq 1$ and $0 \leq j \leq c + k - 1$. Let $V = m + \theta$, where m is a positive integer and $0 \leq \theta < 1$. Then $m = k$ and $\theta = j/(c + k)$.*

Proof. Using the optimal stopping rule defined above we have

$$V = \frac{1}{N}\left(\sum_{i=1}^{\lfloor c+V \rfloor} i + \sum_{i=\lfloor c+1+V \rfloor}^{N} (c + V)\right).$$

Using the representation $V = m + \theta$, this becomes

$$m + \theta = \frac{1}{N}\left(\sum_{i=1}^{c+m} i + \sum_{i=c+1+m}^{N} (c + m + \theta)\right).$$

Subtracting $m + \theta$ from both sides and then multiplying by N, we get

$$0 = \sum_{i=1}^{c+m} (i - m - \theta) + c(N - (c + m)).$$

Thus,

$$cN = \sum_{i=1}^{c+m} (c + m - i) + (c + m)\theta.$$

Plugging in the triangular representation for cN we have $1 + \cdots + (c + k - 1) + j = 1 + \cdots + (c + m - 1) + \theta(c + m)$. Since $0 \leq \theta < 1$, we must have $k = m$ and $j = \theta(c + m)$. ∎

Corollary 1. *In the case of cost c for an extra roll, V is an integer if and only if cN is a triangular number.*

Corollary 2. *In the case $c = 1$, if $N = 1 + \cdots + m + j$ with $0 \le j \le m$, then $V = m + \frac{j}{1+m}$ and the optimal rule is to stop the first time that $X_n \le 1 + m$.*

Here, m is the triangular root of the greatest triangular number less than or equal to N. In particular, if N is triangular, then V is the triangular root of N. For the rest of the paper, for simplicity, we will restrict attention to the case where c=1. Table 1 illustrates the results in Corollary 2.

TABLE 1: The minimal expected loss with $c = 1$ and $N = (1 + \cdots + m) + j$.

N	1	2	3	4	5	6	7	8	9	10
$1 + m$	2	2	3	3	3	4	4	4	4	5
$E(L) = V$	1	$1\frac{1}{2}$	2	$2\frac{1}{3}$	$2\frac{2}{3}$	3	$3\frac{1}{4}$	$3\frac{2}{4}$	$3\frac{3}{4}$	4

In general, the optimal stopping rule stops the first time that $X_n \le 1 + m$. However, if N is triangular and V is an integer, you could also stop when $X_n \le m$ and continue if $X_n = 1 + m$. In the case of the 100 sided die, we see that 91 is the greatest triangular number less than or equal to 100. Its triangular root is 13. Hence, you would stop if $X_n \le 14$. Since $100 = 91 + 9$, we have $\theta = 9/14$ and $V = 13\frac{9}{14}$. We can also write explicit formulas for m and θ in terms of N. We have that if $x(1 + x) = 2N$, then the greatest triangular number less than or equal to N is $\lfloor x \rfloor (1 + \lfloor x \rfloor)/2$. We thus have:

Corollary 3. *We have $V = m + \theta$, where*

$$m = \left\lfloor \frac{-1 + \sqrt{1 + 8N}}{2} \right\rfloor \qquad \text{and} \qquad \theta = \frac{N}{1 + m} - \frac{m}{2}.$$

Distinguishing between optimal stopping times.

Let T be the number of rolls until the game ends. Then the loss to the player is $L = X_T + T - 1$. Many interesting conclusions follow from this for the N-sided die game. Assume that $N = 1 + \cdots + m$ is a triangular number. If T_1 is the first time that $X_n \le 1 + m$, then X_{T_1} is uniform on $\{1, \ldots, 1 + m\}$, and T_1 is geometric with

$$p = \frac{1 + m}{N} = \frac{2}{m}.$$

Thus

$$E(X_{T_1}) = \frac{2 + m}{2} \qquad \text{and} \qquad E(T_1) = \frac{m}{2}.$$

Hence, $V_1 = E(L_1) = m$ as in Corollary 2 to Theorem 1.

If T_2 is the first time that $X_n \le m$, then X_{T_2} is uniform on $\{1, \ldots, m\}$, and T_2 is geometric with

$$p = \frac{m}{N} = \frac{2}{1 + m}.$$

In this case,

$$E(X_{T_2}) = \frac{1 + m}{2} \qquad \text{and} \qquad E(T_2) = \frac{1 + m}{2}.$$

Hence, $V_2 = E(L_2) = m$, the same as above, and confirming earlier statements about these two competing stopping rules. Then we have the question, "Which is better?" One way to decide is to compare their variances. Using the fact that

$$\text{Var } L = \text{Var } X_T + \text{Var } T,$$

we find

$$\text{Var } L_1 = \frac{(1+m)^2 - 1}{12} + \frac{1 - \frac{2}{m}}{4/m^2} = \frac{m^2 - m}{3}.$$

Using T_2, we get

$$\text{Var } L_2 = \frac{m^2 - 1}{12} + \frac{1 - \frac{2}{m+1}}{4/(m+1)^2} = \frac{m^2 - 1}{3}.$$

Thus, T_1 and T_2 lead to the same expected losses, but using T_1 leads to a smaller variance of the loss. Also, $T_1 \leq T_2$. On both counts, T_1 is preferable to T_2. The variance of the loss is the sum of the variances of the stopping time and the variance of the final payment. Stopping when $X_n \leq 1 + m$, rather than $X_n \leq m$, decreases the variance of the stopping time time more than it increases the variance of the final payment.

The duality between the two problems

We now apply these results to the original maximal expected gain problem. If the die roll gives X, then let $Y = N + 1 - X$. Then Y is also uniformly distributed on $1, 2, \ldots, N$ and can be treated as the outcome of the roll of an N-sided die. We can play the maximal expected gain problem using the observations $Y_1, Y_2, \ldots$ as rolls of a die and the minimal expected loss problem using observations $X_1, X_2, \ldots$ at the same time.

If you stop at time T, and if you denote by G the reward minus the payments, then we have

$$G = Y_T - T + 1 = N + 1 - X_T - T + 1 = N + 1 - L.$$

The optimal T that maximizes the expected gain also minimizes the expectation of $X_T + T - 1 = L$, that is, the loss in the previous problem. Thus, if you stop when $X_n \leq 1 + m$ in the minimal expected loss problem, then you stop when

$$Y_n = N + 1 - X_n \geq N - m$$

in the maximum expected gain problem. The distribution of the stopping time for the maximal expected game problem is the same as for the minimal expected loss problem. The final gain would equal $N + 1 - L$, where L is the loss. We would also have

$$E(G) = N + 1 - E(L) = N - m + (1 - \theta)$$

and $\text{Var } G = \text{Var } L$.

These formulas are simplest when $N = 1 + \cdots + m$ is a triangular number and $V = m$ is an integer. In this case, we stop if $X_n \leq 1 + m$ or

$$Y_n \geq N - m = 1 + \cdots + (m - 1),$$

the greatest triangular number less than N. Table 2 illustrates that the pattern for the expected maximal gain is harder to discern than the pattern for the minimal expected loss.

TABLE 2: The maximal expected gain with $c = 1$ and $N = (1 + \cdots + m) + j$.

N	1	2	3	4	5	6	7	8	9	10
$N - m$	0	1	1	2	3	3	4	5	6	6
$E(G) = V$	1	$1\frac{1}{2}$	2	$2\frac{2}{3}$	$3\frac{1}{3}$	4	$4\frac{3}{4}$	$5\frac{2}{4}$	$6\frac{1}{4}$	7

In the important case $N = 100$ and $m = 13$, we showed for the minimal expected loss problem earlier that we stop if $X_n \le 14$ and $V = 13\frac{9}{14}$. This implies that for the maximal expected gain problem we stop if $Y_n \ge 87$ and $E(G) = 87\frac{5}{14} = 87.367$. This last value agrees with the approximate solutions given on the internet and requires much less work.

The limiting distributions of the losses and gains.

The exact distributions of the losses and gains are not difficult, but the general formulas are a little messy. We therefore limit attention to the loss distribution in playing the game with a 6-sided die. The solution pattern illustrates what happens more generally. We stop the first time T that $X_n \le 4$. The loss is then $L = T + X_T - 1$, where X_T is uniform on $\{1, 2, 3, 4\}$ and is independent of T, which is geometric ($p = 2/3$). We then have, for example,

$$\begin{aligned} P(L = 2) &= P(T = 1)P(X_T = 2) + P(T = 2)P(X_T = 1) \\ &= \left(1 + \left(\frac{1}{3}\right)\right)\left(\frac{2}{3}\right)\left(\frac{1}{4}\right) = \frac{2}{9}. \end{aligned}$$

We also have

$$\begin{aligned} P(L = 4 + j) &= [P(T = j + 1) + P(T = j + 2) \\ &\qquad + P(T = j + 3) + P(T = j + 4)](1/4) \\ &= \left(\frac{1}{3}\right)^j \left(\frac{2}{3}\right)\left(1 + \left(\frac{1}{3}\right) + \left(\frac{1}{3^2}\right) + \left(\frac{1}{3^3}\right)\right)\left(\frac{1}{4}\right) \\ &= \left(\frac{1}{3}\right)^j \left(\frac{20}{81}\right). \end{aligned}$$

Similar arguments give us Table 3.

TABLE 3: The distribution of the loss for a six-sided die.

j	1	2	3	4	$4 + j$
$P(L = j)$	1/6	2/9	13/54	20/81	$(1/3)^j(20/81)$

We now move to the limiting case. Let L_N be the loss when using the optimal strategy in the case of an N-sided die. Let m_N be the optimal m for an N-sided die and T_N, the optimal stopping time. We consider the limiting distributions of

$$\frac{X_{T_N}}{1 + m_N}, \qquad \frac{T_N}{1 + m_N}, \qquad \text{and} \qquad \frac{L_N}{1 + m_N}.$$

It is easily seen that $X_{T_N}/(1+m_N)$ converges in distribution to U, where U is uniformly distributed over [0,1]. Also, T_N is geometric with

$$p = \frac{1+m_N}{N} \sim \frac{2}{1+m_N}.$$

It follows that $T_N/(1+m_N)$ converges in distribution to W, which is exponential with mean 1/2. That is, $P(T_N > k) = (1-p)^k$. Thus,

$$P\left(\frac{T_N}{1+m_N} > w\right) = P\left(T_N > \lfloor(w(1+m_N)\rfloor\right)$$
$$= (1 - \frac{1+m_N}{N})^{\lfloor(w(1+m_N)\rfloor} \to e^{-2w}.$$

Using the independence of X_{T_N} and T_N, we have proved the theorem:

Theorem 2. *If L_N is the loss using the optimal strategy with an N-sided die, then as $N \to \infty$, we have*

$$P\left(\frac{L_N}{1+m_N} \le s\right) \to P(U+W \le s) = F(s),$$

where U is uniform on $[0, 1]$ *and W is exponential* $(\lambda = 2)$ *and independent of U.*

Corollary 4. *For $s \le 1$,*

$$P\left(\frac{L_N}{1+m_N} \le s\right) \to s - \frac{1}{2}(1-e^{-2s}) = F(s).$$

For $s > 1$,

$$P\left(\frac{L_N}{1+m_N} \le s\right) \to 1 - \frac{e^{-2s}}{2}(e^2-1) = F(s).$$

Proof. For $s \le 1$,

$$P\left(U+W \le s\right) = \int_0^s \int_0^{s-u} 2e^{-2w}\, dw\, du.$$

For $s > 1$,

$$P\left(U+W \le s\right) = \int_0^1 \int_0^{s-u} 2e^{-2w}\, dw\, du.$$

■

If we consider the maximal gain problem instead of the minimal loss problem, then we may assume $Y_n = N+1-X_n$ as before. We then have $N+1-G_N = L_N$, which leads to:

Corollary 5. *If G_N is the gain using the optimal strategy with an N-sided die, then*

$$P\left(\frac{N+1-G_N}{1+m_N} \le s\right) \to P(U+W \le s) = F(s).$$

This brings us back to the 100-sided fair die optimal stopping game. Using the approximations in Corollary 5, we have

$$P(101 - G \leq 14s) \approx F(s).$$

This means $P(G \geq 101) \approx 0$. This is not surprising since $G \leq 100$ in this case. On the other hand,

$$P(G \geq 73) \approx 1 - F(2) = .94.$$

Thus there is a high probability that the gain will be between 73 and 100.

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Summary. A fair die with sides numbered from one to N is rolled. You can either receive the value shown or else pay one and roll again. If you do not like the second roll, you can pay another one and roll again. This can be repeated as many times as you wish. The questions are, When should you roll again? and What is the expected return from playing this game?" This is the N-sided die optimal stopping game. It has become a popular problem on the internet in the case N=100 with several solutions given. There is a weakness in the solutions, however, in that they use numerical approximations. In this paper we find exact formulas for the solution for arbitrary N and show that the stopping rules and expected returns form an interesting pattern as N increases that is related to the sequence of triangular numbers.

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The Generalized Sum of Remainders Map and Its Fixed Points

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Over the last few decades, number theory has been used as the main tool for studying the iteration of some interesting maps. Among others (but not discussed here), there is Collatz's famous $3n + 1$ map: a self-map on the set of natural numbers which halves the argument if it is even and computes three times the argument plus one if it is odd. A famous open conjecture states that every orbit of this map eventually reaches one. Similarly, mathematicians have studied Bulgarian solitaire: a self-map on the space of finite integer multisets, some of whose orbits are proved to be eventually constant [**4**, **5**]. In the same spirit, in this paper, we address the iteration of another number-theoretic map which produces intriguing orbits.

Take a finite nonempty set of positive integers, say $K = \{1, 3, 4\}$. Starting with any positive integer, say 11, produce a new number by computing the sum of the remainders upon dividing this integer by every element of K. Since dividing 11 by 1, by 3, and by 4 leaves remainders 0, 2, and 3, respectively, our new number is $0 + 2 + 3 = 5$. Now, apply the same process into this new number. That is, divide it by every element of K and compute the sum of the remainders. Doing this repeatedly, we obtain the following sequence: $(11, 5, 3, 3, 3, 3, 3, \dots)$. In this case, the sequence becomes constant. However, if we start with, for example, 8, we obtain the sequence $(8, 2, 4, 1, 2, 4, 1, \dots)$ which turns out to be periodic of period 3. The question is: Starting with any positive integer, is there a way to predict the behavior of the resulting sequence?

More formally, let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For any finite nonempty set K of positive integers, we define a map $\rho_K : \mathbb{N}_0 \to \mathbb{N}_0$ by

$$\rho_K(x) = \sum_{k \in K} (x \bmod k) \tag{1}$$

for every $x \in \mathbb{N}_0$. We are interested in the dynamics of this map, that is, the behavior of the sequence $(x, \rho_K(x), \rho_K(\rho_K(x)), \dots)$ for any given *initial value* $x \in \mathbb{N}_0$. This sequence is referred to as the *orbit* of x.

Some maps similar to ρ_K are found in the literature. For instance, Spivey [**6**] discusses a map which computes the sum of the remainders upon dividing any positive integer by every positive integer not exceeding it. Therefore, in this map the set K depends on the argument, that is, $K = \{1, 2, \dots, x\}$. Boju and Funar [**1**, page 76] eliminate this dependence by using $K = \{1, 2, \dots, m\}$, where m is a fixed positive integer. Our map (1) can be viewed as a further generalization, which consists of replacing K with any fixed finite nonempty set of positive integers. The dynamics of our map

Math. Mag. **94** (2021) 288–295. doi:10.1080/0025570X.2021.1951565
MSC: 11A25, 37P99, 11B50

ρ_K, therefore, may also share some similarities with the maps found in the literature, and so our study of ρ_K might also be beneficial towards understanding them in the perspective of dynamical systems.

To illustrate a typical behavior of our map, let us now return to our choice $K = \{1, 3, 4\}$. We can compute

$$(\rho_K(x))_{x=0}^{\infty} = (0, 2, 4, 3, 1, 3, 2, 4, 2, 1, 3, 5, 0, 2, 4, 3, 1, 3, 2, 4, 2, 1, 3, 5, \dots),$$

from which we make the following two important observations.

- Notice that $\rho_K(x + 12) = \rho_K(x)$ for every $x \in \mathbb{N}_0$. That is, the sequence $(\rho_K(x))_{x=0}^{\infty}$ is periodic of period $12 = \operatorname{lcm}\{1, 3, 4\}$. Therefore, it suffices to restrict our attention to initial values in $\{0, 1, \dots, 11\}$.
- For every $x \in \mathbb{N}_0$, since $x \bmod 1 \in \{0\}$, $x \bmod 3 \in \{0, 1, 2\}$, and $x \bmod 4 \in \{0, 1, 2, 3\}$, we have

$$\rho_K(x) = (x \bmod 1) + (x \bmod 3) + (x \bmod 4) \in \{0, 1, 2, 3, 4, 5\}.$$

Having computed the images of all initial values of interest, we can now give a complete description of all their orbits by constructing the directed graph shown in Figure 1, having vertex set $\{0, 1, \dots, 11\}$ and edge set

$$\{(x, \rho_K(x)) : x \in \{0, 1, \dots, 11\}\}.$$

We call this graph the *orbit graph* of the map ρ_K.

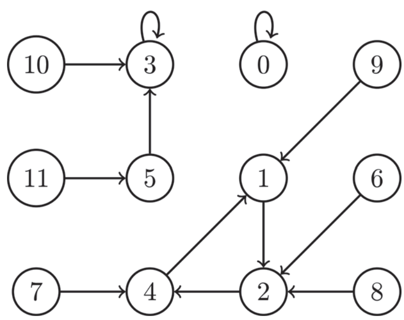

Figure 1 The orbit graph of ρ_K, where $K = \{1, 3, 4\}$.

The graph shows the existence of three disjoint sets which form cycles, namely $\{0\}$, $\{1, 2, 4\}$, and $\{3\}$. In other words, 0 and 3 are periodic points of prime period 1, i.e., fixed points, whereas 1, 2, and 4 are periodic points of prime period 3 forming a 3-cycle. Different choices of the set K could give periodic points of other periods, as shown in Figures 2 and 3.

For any finite $K \subset \mathbb{N}$, it is clear that ρ_K takes only finitely many possible values, so it is obvious that every initial condition is eventually periodic. However, the existence of periodic points of various specified periods is not at all obvious. Given any finite nonempty set K of positive integers and any $n \in \mathbb{N}$, one could question, for instance, whether the map ρ_K possesses periodic points of prime period n.

We would agree that answering this question demands a thorough analysis of the number-theoretic properties of the set K, as well as that of the initial value x. This paper aims to begin this analysis for the simplest type of periodic points, namely fixed points.

The general setup

We briefly restate our general setup. For the precise definitions of dynamical and number-theoretic terminology used in this paper, we refer the reader to Devaney [3] and Burton [2], respectively.

Let $K \subset \mathbb{N}$ be finite and nonempty. Define the map $\rho_K : \mathbb{N}_0 \to \mathbb{N}_0$ by the rule

$$\rho_K(x) = \sum_{k \in K} (x \bmod k)$$

for every $x \in \mathbb{N}_0$. First, we prove the general versions of our two observations in the introduction.

Proposition 1. *The sequence* $(\rho_K(x))_{x=0}^{\infty}$ *is periodic of period* $\operatorname{lcm} K$.

Proof. Notice that

$$\begin{aligned}\rho_K(x + \operatorname{lcm} K) &= \sum_{k \in K} [(x + \operatorname{lcm} K) \bmod k] \\ &= \sum_{k \in K} (x \bmod k) = \rho_K(x),\end{aligned}$$

where we have used the fact that k evenly divides $\operatorname{lcm} K$ for every $k \in K$. ∎

Proposition 2. *For every nonnegative integer* x*,*

$$\rho_K(x) \in \left\{0, 1, \ldots, \sum_{k \in K} k - |K|\right\}.$$

Proof. For every nonnegative integer x, we have $x \bmod k \in \{0, 1, \ldots, k-1\}$ for all $k \in K$, so $\rho_k(x) = \sum_{k \in K} (x \bmod k)$ belongs to the required set. ∎

These two propositions allow us to restrict the domain and the codomain of ρ_K, so that we can formally define it as follows:

Definition 1. Let $K \subset \mathbb{N}$ be finite and nonempty, and let

$$I_K := \{0, 1, \ldots, \operatorname{lcm} K - 1\} \quad \text{and} \quad S_K := \left\{0, 1, \ldots, \sum_{k \in K} k - |K|\right\}.$$

The *generalized sum of remainders map* over K is defined as the map $\rho_K : I_K \cup S_K \to S_K$ given by

$$\rho_K(x) = \sum_{k \in K} (x \bmod k).$$

Looking at our previous example, one might be tempted to define ρ_K as a map from I_K to S_K. However, notice that we can have $I_K \subset S_K$ (see, for example, Figure 3), in which case the map ρ_K is not well-defined for every element in $S_K \backslash I_K$. Hence, the domain of ρ_K is set to be $I_K \cup S_K$.

Fixed points

It is clear that 0 is a fixed point of ρ_K for any finite nonempty $K \subset \mathbb{N}$. If $|K| = 1$, it is easy to see that every point in $I_K \cup S_K$ is fixed. Therefore, we only need to consider the case $|K| \geq 2$. We now give a sufficient condition for the existence of nonzero fixed points. We define the notation $L_K := \operatorname{lcm}(K \backslash \{\max K\})$. That is, L_K is the least common multiple of the numbers in the set $K \backslash \{\max K\}$.

Theorem 1. *Let $|K| \geq 2$, and let $L_K < \max K$. Then every nonnegative multiple of L_K less than* $\max K$ *is a fixed point of ρ_K.*

Proof. Let $x < \max K$ be an arbitrary nonnegative multiple of L_K. Then $k \mid x$, which implies $x \bmod k = 0$, for all $k \in K \backslash \{\max K\}$. Consequently,

$$\begin{aligned}
\rho_K(x) &= \sum_{k \in K} (x \bmod k) \\
&= \sum_{k \in K \backslash \{\max K\}} (x \bmod k) + [x \bmod (\max K)] \\
&= 0 + [x \bmod (\max K)] \\
&= x \bmod (\max K) = x,
\end{aligned}$$

since $x < \max K$. This means that x is a fixed point of ρ_K. ■

Theorem 1 tells us that under the given condition, every nonnegative multiple of L_K less than $\max K$ is a fixed point of ρ_K. However, it does not guarantee that there is no other fixed point which is not a multiple of L_K. As an example, in the case of $K = \{3, 6, 7\}$, this theorem tells us that every nonnegative multiple of $\operatorname{lcm}\{3, 6\} = 6$ less than $\max\{3, 6, 7\} = 7$ is a fixed point of ρ_K. These multiples are 0 and 6. However, one can easily verify that 11 is also a fixed point of ρ_K. The orbit graph of ρ_K is displayed in Figure 2.

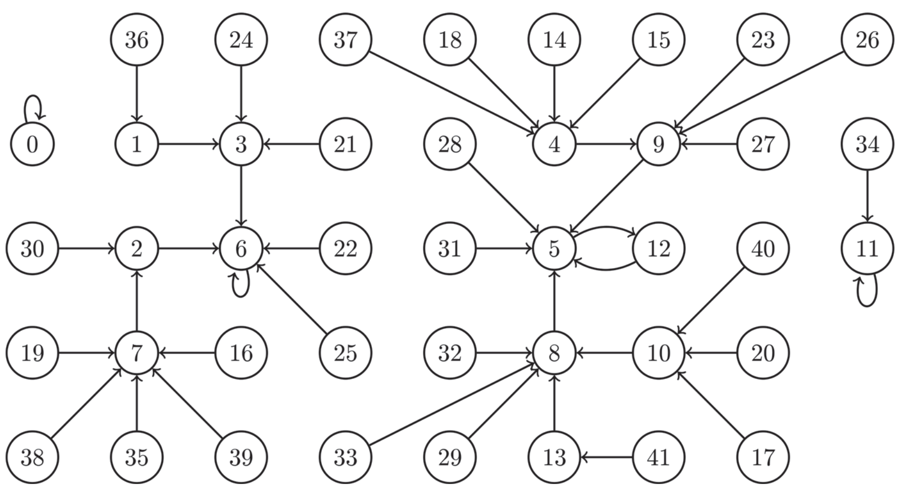

Figure 2 The orbit graph of ρ_K, where $K = \{3, 6, 7\}$.

If the given condition is violated, then a nonzero fixed point may or may not exist. In some cases, it does. For instance, Boju and Funar's map [1, page 76] with

$K = \{1, 2, 3, 4, 5\}$, has $L_K = 12 > 5 = \max K$, and yet has 7 as a fixed point [1, page 76]. In this case, this fixed point is not a multiple of L_K. In such a situation we observe that the following holds:

Proposition 3. *Let $|K| \geq 2$, and let $L_K > \max K$. Let x be a nonzero fixed point of ρ_K which is not a multiple of L_K. Then*

(1) $x \geq \max K$, *and*

(2) $x = \max K$ *if and only if* $x \in K \cap I_{K\setminus\{x\}}$ *and x is a fixed point of* $\rho_{K\setminus\{x\}}$.

Proof. To prove (1), suppose for a contradiction that $x < \max K$. Then $x \bmod (\max K) = x$. Since x is a fixed point of ρ_K, we have $\rho_K(x) = x$, that is,

$$\sum_{k\in K}(x \bmod k) = x$$

$$\sum_{k\in K\setminus\{\max K\}}(x \bmod k) + [x \bmod (\max K)] = x$$

$$\sum_{k\in K\setminus\{\max K\}}(x \bmod k) = 0.$$

Since $x \bmod k \geq 0$ for every $k \in K\setminus\{\max K\}$, we must have $x \bmod k = 0$ for every $k \in K\setminus\{\max K\}$. This means that $k \mid x$ for every $k \in K\setminus\{\max K\}$. It follows that $L_K \mid x$, which is a contradiction.

The necessity part of (2) is a direct consequence of (1) because $x \in K$ and $x \geq \max K$ implies $x = \max K$. To prove the sufficiency, suppose $x = \max K$. Clearly, $x \in K$. Also, $0 < x \leq L_K - 1$, so $x \in I_{K\setminus\{x\}}$. Therefore, x belongs to the domain of $\rho_{K\setminus\{x\}}$. Since x is a fixed point of ρ_K, we have $\rho_K(x) = x$, and therefore

$$\rho_{K\setminus\{x\}}(x) = \sum_{k\in K\setminus\{x\}}(x \bmod k) = \sum_{k\in K\setminus\{x\}}(x \bmod k) + (x \bmod x)$$
$$= \sum_{k\in K}(x \bmod k) = \rho_K(x) = x.$$

This proves that x is a fixed point of $\rho_{K\setminus\{x\}}$. ■

In Figure 1, we can see that the trivial fixed point 0 appears as an isolated vertex. This prompts the following definition:

Definition 2. A fixed point of ρ_K is said to be *isolated* if it is not the image of any element of $I_K \cup S_K$ other than itself.

In general, the trivial fixed point is not always isolated. For example, take $K = \{1, 3, 9\}$. Here we have $I_K = \{0, 1, \ldots, 8\}$ and $S_K = \{0, 1, \ldots, 10\}$. The orbit graph of ρ_K in Figure 3 shows that $\rho_K(9) = 0$.

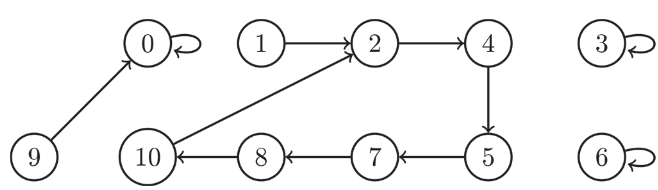

Figure 3 The orbit graph of ρ_K, where $K = \{1, 3, 9\}$.

The following condition determines when the trivial fixed point is isolated:

Proposition 4. *The trivial fixed point of the map ρ_K is isolated if and only if $S_K \subseteq I_K$.*

Proof. First, suppose $S_K \subseteq I_K$. Since for every $x \in I_K \cup S_K = I_K$ we have $x < \operatorname{lcm} K$, it follows that $\rho_K(x) = 0$ if and only if $x = 0$. Conversely, suppose $S_K \supset I_K$. Then $\operatorname{lcm} K \in S_K = I_K \cup S_K$, so we can compute $\rho_K(\operatorname{lcm} K) = 0$, proving that the trivial fixed point is not isolated. ■

One might question whether nonzero isolated fixed points exist. It is easy to show that 1 is a nonzero isolated fixed point of ρ_K, where $K = \{1, 2\}$. However, determining a condition for their existence is not a straightforward task. Tables displaying the list of nonzero isolated fixed points for various choices of K appear in an appendix to the online version of this article.

Pairwise relatively prime set

We shall prove that in the special case in which the integers in K are pairwise relatively prime, the only fixed points of ρ_K are those given by Theorem 1 (as seen in the example in the introduction). Before doing so, we first prove the following lemma:

Lemma 1. *Let $k \in \mathbb{N}$, and let $x_1, x_2, \ldots, x_k \in \mathbb{N}$ be pairwise distinct. Then*

$$x_1 x_2 \ldots x_k \geq x_1 + x_2 + \ldots + x_k - k. \tag{2}$$

Proof. First we prove that for any distinct $a, b \in \mathbb{N}$ with $a, b \geq 2$ we have

$$ab \geq a + b. \tag{3}$$

Without loss of generality, let $a \geq b$. Then $ab \geq a \cdot 2 = a \cdot (1 + 1) = a + a \geq a + b$, as required. Now we prove inequality (2). Since it is trivially true for $k = 1$, let $k \in \mathbb{N}$ with $k \geq 2$, and let $x_1, x_2, \ldots, x_k \in \mathbb{N}$ be pairwise distinct. If $x_i \geq 2$ for every $i \in \{1, 2, \ldots, k\}$, then applying inequality (3) repeatedly yields

$$\begin{aligned} x_1 x_2 \ldots x_{k-2} x_{k-1} x_k &= (x_1 x_2 \ldots x_{k-2} x_{k-1})\, x_k \\ &\geq (x_1 x_2 \ldots x_{k-2})\, x_{k-1} + x_k \\ &\geq x_1 x_2 \ldots x_{k-2} + x_{k-1} + x_k \\ &\vdots \\ &\geq x_1 + x_2 + \ldots + x_k \\ &\geq x_1 + x_2 + \ldots + x_k - k, \end{aligned}$$

as desired. Otherwise, there exists $i \in \{1, 2, \ldots, k\}$ such that $x_i < 2$, implying that $x_i = 1$. Then $x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{k-1}, x_k$ are all ≥ 2 since $x_1, x_2, \ldots, x_k$ are pairwise distinct positive integers. Therefore,

$$\begin{aligned} x_1 x_2 \ldots x_k &= x_1 x_2 \ldots x_{i-1} \cdot 1 \cdot x_{i+1} \ldots x_{k-1} x_k \\ &= x_1 x_2 \ldots x_{i-1} x_{i+1} \ldots x_{k-1} x_k \\ &\geq x_1 + x_2 + \ldots x_{i-1} + x_{i+1} + \ldots + x_{k-1} + x_k \\ &= x_1 + x_2 + \ldots x_{i-1} + 1 + x_{i+1} + \ldots + x_{k-1} + x_k - 1 \\ &= x_1 + x_2 + \ldots x_{i-1} + x_i + x_{i+1} + \ldots + x_{k-1} + x_k - 1 \end{aligned}$$

$$\geq x_1 + x_2 + \ldots + x_k - k,$$

where the first inequality follows from applying inequality (3) repeatedly, as above. ■

The following corollary of Lemma 1 is easily derived.

Corollary 1. *For any finite nonempty subset K of pairwise relatively prime positive integers, we have $|I_K| \geq |S_K| - 1$.*

Proof. By Lemma 1, we have

$$|I_K| = \operatorname{lcm} K = \prod_{k \in K} k \geq \sum_{k \in K} k - |K| = \left(\sum_{k \in K} k - |K| + 1\right) - 1 = |S_K| - 1,$$

as desired. ■

We now state and prove our promised result.

Theorem 2. *Let K be a finite set of pairwise relatively prime positive integers with $|K| \geq 2$ and such that $\prod_{k \in K \setminus \{\max K\}} k < \max K$. Then the fixed points of ρ_K are precisely the nonnegative multiples of $\prod_{k \in K \setminus \{\max K\}} k$ less than $\max K$.*

Proof. Since in this case we have $L_K = \prod_{k \in K \setminus \{\max K\}} k$, then by Theorem 1, every nonnegative multiple of $\prod_{k \in K \setminus \{\max K\}} k$ less than $\max K$ is a fixed point of ρ_K. Now we prove the converse. Let x be an arbitrary fixed point of ρ_K, then $x = \rho_K(x)$, that is, $x = \sum_{k \in K} (x \bmod k)$. Equivalently,

$$x - [x \bmod (\max K)] = \sum_{k \in K \setminus \{\max K\}} x \bmod k.$$

Since the left-hand side is a multiple of $\max K$, then so is the right-hand side. But the right-hand side is a nonnegative integer not exceeding

$$\sum_{k \in K \setminus \{\max K\}} k - (|K| - 1) \leq \prod_{k \in K \setminus \{\max K\}} k < \max(K),$$

where the first inequality follows from Lemma 1. This forces $x - [x \bmod (\max(K))] = 0$, i.e.,

$$0 = \sum_{k \in K \setminus \{\max(K)\}} (x \bmod k).$$

Now each summand on the right hand side is nonnegative, so they must all be zero, implying that every element of $K \setminus \{\max(K)\}$ divides x. It follows that x is a multiple of $L_K = \prod_{k \in K \setminus \{\max K\}} k$, as desired. ■

In the special case where $|K| = 2$, the hypothesis of Theorem 2 is always satisfied, and therefore we can easily deduce the following corollary:

Corollary 2. *Let K be a set of two relatively prime positive integers. Then the map ρ_K has exactly $\left\lceil \frac{\max(K)}{\min(K)} \right\rceil$ fixed points, specifically, all nonnegative multiples of $\min(K)$ less than $\max(K)$.*

This corollary gives a complete description of fixed points of ρ_K for $|K| = 2$. One can also see that if $1 \notin K$, then we have $I_{K \cup \{1\}} = I_K$, $S_{K \cup \{1\}} = S_K$, and indeed, $\rho_{K \cup \{1\}} = \rho_K$. Therefore, the above corollary also extends to the case in which $|K| = 3$, $1 \in K$, and $\gcd(K \setminus \{1\}) = 1$.

Conclusion

We have introduced the generalized sum of remainders map ρ_K, given by (1). In the case $\operatorname{lcm}(K\backslash\{\max(K)\}) < \max(K)$, we have established a family of fixed points (Theorem 1) and proved that these are the only fixed points if the elements of K are pairwise relatively prime (Theorem 2). We have also derived some results in the opposite case (Proposition 3) as well as a necessary and sufficient condition for the trivial fixed point to be isolated (Proposition 4). The study can be continued by considering the following open questions:

1. Is there a necessary and sufficient condition for the existence of nonzero fixed points, and more generally, nonzero periodic points?
2. Similarly, is there a firm condition for a nonzero fixed point, and more generally, for a periodic orbit, to be isolated?
3. Does the number of iterations needed for an orbit to reach one of the periodic points depend regularly on its initial value?
4. Can we further generalise to the case where K is a multiset, rather than a set?

Acknowledgments The authors thank the Indonesia Endowment Fund for Education (LPDP) for the financial support received during the writing of this paper.

Appendix

The online version of this article contains an appendix with tables listing the fixed points of the map ρ_K for every nonempty subset K of $\{7, 12, 13, 17, 23, 25\}$.

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Summary. We generalize the sum of remainders map by replacing the set of summation indices with an arbitrary finite nonempty set of positive integers. The iteration of this map produces interesting dynamical behavior, which we explore. In particular, we discuss the existence of nonzero fixed points and some of their properties.

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On the Multiplicity of Polyabolos and Tangrams with Four-Fold Symmetry

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Many of us fondly remember playing Tangrams [1, 6, 10], where the goal is to pack seven special tiles into a given pattern specified only by its boundary. There are several such silhouette puzzles [10], including the fourteen-tile "loculus" of Archimedes and the seven-tile "Sei Shonagon Chie no Ita" from Japan. Two tile sets and example arrangements are shown in Figure 1.* The appeal is universal, and there are rich opportunities for exploration [3, 4] and pedagogical use [12]. So it is not surprising that Tangrams have attracted the attention of mathematicians: Scott [8] and Wang and Hsiung [11] proved that exactly 13 convex polygons can be formed with the seven Tangram tiles. Read [7] showed that 4,842,205 Tangram patterns exist that are *snug*, meaning that the pattern has no holes and the tiles abut each other along an entire edge (or half edge in the case of the large triangles). He also showed that the number increases to 5,583,516 if holes are allowed. Graber et al.[5] showed that 5,520 "star" Tangrams exist, where all seven pieces meet at one point.

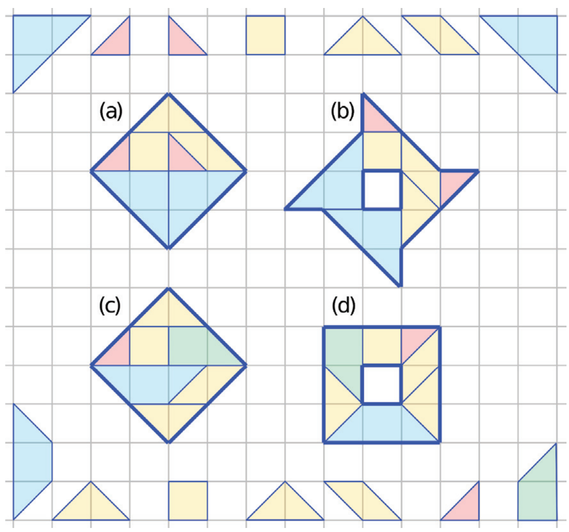

Figure 1 Tangram tiles along top, Sei Shonagon Chie no Ita tiles along bottom, all color-coded by area, and two example patterns that can be formed by each. These patterns all have *four-fold rotational symmetry*: rotation by ninety degrees leaves the boundary unchanged. Here, the boundaries are indicated by thick line segments. Patterns (a,c,d) also have *reflection symmetry*: reflection in a mirror leaves the boundary unchanged. Pattern (b) is a *chiral* pinwheel: reflection in a mirror changes the boundary between left- and right-handed versions that cannot be superimposed by rotation.

Math. Mag. **94** (2021) 296–301. doi:10.1080/0025570X.2021.1952040
MSC: primary 05B50, secondary 52B45

*The online version of this article features color diagrams that might be easier to read.

The arguments presented by Read [7] and Wang and Hsiung [11] rely on the fact that Tangram tiles can be dissected into congruent right isosceles triangles joined along equal-length edges; such shapes are now called *polyabolos*. As can be seen in Figure 1 from the superposition of the patterns onto a square grid, the Tangram and Sei Shonagon Chie no Ita tile sets can both be dissected into 16 base units. Observe there that the seven Tangram tiles consist of two size-1 base unit triangles (shown in red), three different size-2 polyabolos (yellow), and two size-4 triangles (light blue). In contrast, the Sei Shonagon Chie no Ita tiles consist of one base unit (red), four size-2 polyabolos (yellow), one size-3 polyabolo (green), and one size-4 polyabolo (blue). If such tile sets are arranged with snug abutments, then the pattern is a size-16 polyabolo.

Wang and Hsiung showed that there exist exactly twenty size-16 convex polyabolos, thirteen of which can be constructed using Tangram tiles. More recently Fox-Epstein et al. [2] showed that Sei Shonagon Chie no Ita tiles can form sixteen convex polygons, and that exactly four seven-piece dissections of the square exist that can form nineteen out of the twenty size-16 convex polyabolos. Integer sequence A006074 [9] gives the number $a(n)$ of distinct size-n polyabolos as

$$\{1, 3, 4, 14, 30, 107, 318, \ldots\},$$

not counting reflections as distinct. Since $a(2) = 3$, Tangrams use all three of the size-2 polyabolos. Also, since $a(4) = 14$, the fourteen size-4 polyabolos shown in Figure 2 cover all distinct possibilities, not counting reflections as unique; this result is needed below. The value of $a(16)$ is not yet known, but it extrapolates exponentially versus n to roughly thirty million. Read's result implies that about $1/5$ of these can be formed by Tangrams.

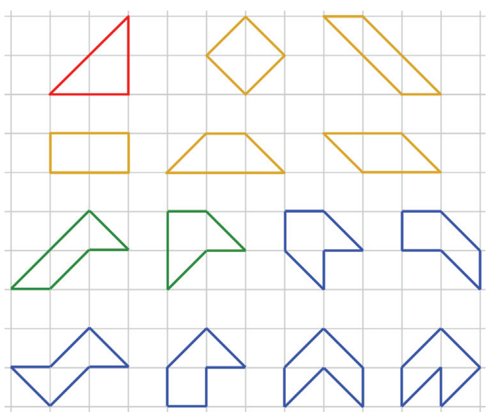

Figure 2 The fourteen size-4 polyabolos, color-coded by the number of edges.

While convexity and snugness are interesting properties for categorizing Tangrams, another aesthetic way is by their symmetry under reflection or rotation. For example those in Figure 1 all have *four-fold rotational symmetry*, meaning that rotation by multiples of ninety degrees leaves the boundary unchanged. It is natural to wonder: Do any other such four-fold Tangrams exist? In fact, there are only two. This is proved, below, using an approach similar in spirit to Wang and Hsiung [11]. I first establish some lemmas enabling the construction of a small number of candidate patterns, and then I test which of these may be formed from the seven Tangram tiles.

Construction of four-fold Tangrams

To begin, it is conceivable that some four-fold Tangram patterns consist of disconnected tiles that do not touch, or touch only at points. One such possibility is four disconnected congruent size-4 shapes, arrayed at multiples of ninety degrees with respect to one another. Since there are two size-4 triangular tiles, which may not be physically cut, the remaining five tiles would be required to form two additional size-4 triangles. This cannot be done. Other disconnected possibilities are a central four-fold shape of size $C = 4, 8$, or 12, surrounded by four congruent shapes of size $S = 3, 2$, or 1 respectively (subject to $C + 4S = 16$). Case $C = 4$ fails because a size-4 triangular tile can be neither the central shape (it is not four-fold) nor a surrounding shape (it is too large). Case $C = 8$ requires that the two triangular tiles form a central square, but fails because the remaining five tiles cannot form four congruent size-2 shapes. Case $C = 12$ fails because there are only two size-1 triangular tiles. All possibilities with disconnected shapes are thus eliminated. This establishes Lemma 1:

Lemma 1. *All four-fold Tangrams are simply- or multiply-connected planar solids.*

Lemma 1 implies that if a four-fold Tangram is literally dissected into four congruent shapes, then at least one tile must be physically cut. Otherwise, intact groupings of tiles could be slid apart into four separated congruent shapes. A physical cut can be made along an edge or diagonal of the square grid underlying the tile, so that it is split into two polyabolos. The corresponding cuts in the other quadrants of the dissection are mutually oriented at ninety degrees and must also split tiles into polyabolos or be along snug abutments of tiles. Therefore, tiles in a quadrant must solidly span and snugly abut two bounding cuts oriented at ninety degrees. If less than three tiles are involved, as happens in at least one quadrant since there are only seven tiles, then they must be aligned on the same square grid. Such alignment is required by symmetry in the other quadrants. This establishes Lemma 2:

Lemma 2. *All four-fold Tangrams are size-*16 *polyabolos.*

Since a four-fold Tangram is a size-16 polyabolo, it may be dissected into four congruent size-4 polyabolos each rotated ninety degrees with respect to two snugly abutting neighbors. As noted above, there are exactly fourteen distinct size-4 polyabolos, all shown in Figure 2. Therefore, all four-fold size-16 polyabolos may be constructed by testing all the ways to snugly assemble four copies of each of the Figure 2 shapes into a four-fold polyabolo. It is not onerous to try all possibilities and thereby establish Lemma 3:

Lemma 3. *There exist exactly ten four-fold size-16 polyabolos (not counting reflections as distinct), as depicted in Fig. 3.*

In principle these could have been mined from the thirty million or so $n = 16$ polyabolos, were they known.

The final step is to play the Tangram game of trying to pack all seven tiles into Figure 3 silhouettes. Possible arrangements can be systematically tested by (1) placing the first large triangle into all possible locations on the underlying grid, (2) for each of these, placing the second large triangle into all available locations on the underlying grid, (3) for each of these, placing the square into all available locations on the underlying grid, and so-on for the remaining tiles. We thereby establish the main proposition:

Proposition 1. *There exist exactly two four-fold Tangrams (not counting reflections as distinct).*

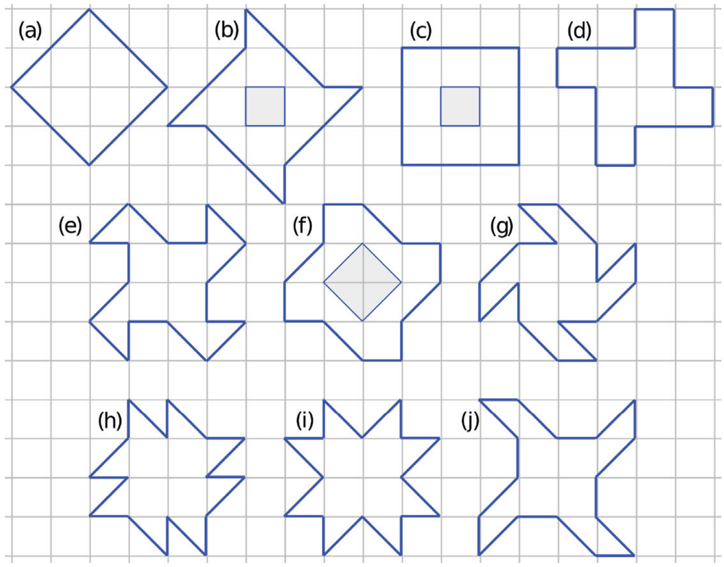

Figure 3 The ten size-16 polyabolos with four-fold rotational symmetry. Only three have reflection symmetry. The other seven are chiral, with left- and right-handed versions interchanged by reflection.

The first of these is the iconic simply-connected square of Figure 1a. The second is the pinwheel with square hole of Figure 1b. This pinwheel does not appear in any publication I have examined, of which Elffers and Schuyt [**1**], Read [**6**], and Slocum [**10**] are particularly comprehensive and scholarly, but it can be found on the internet.

With the same approach, we find that Sei Shonagon Chie no Ita tiles can form exactly two four-fold patterns (Figures 1c, d), both well-known. Thus, in both Tangram and Sei Shonagon Chie no Ita the classical square is the only four-fold and simply-connected pattern using all pieces. Similarly, we find that the Fox-Epstein, Katsumata, and Uehara dissections {a, b, c, d} can respectively make {3, 1, 3, 2} four-fold polyabolos: All can form the solid square, two can form the square with a square hole, two can form the top-right pattern in Figure 3, and one can form the pinwheel in Figure 1b, which thus deserves to be better known as an iconic Tangram pattern. These dissections can actually form an infinite number of four-fold patterns that are not polyabolos, since each set of tiles can form four disconnected size-4 parallelograms.

Other four-fold polyabolos

Four-fold polyabolos of different sizes may be constructed, per above, by four-fold assembly of known polyabolos. The number $a(n)$ of existing four-fold polyabolos of size-n is $a(n) = 0$ for n odd, of course. We find

$$a(n) = \{1, 1, 1, 1, 3, 4, 6, 10, 18\}$$

for

$$n = \{2, 4, 6, 8, 10, 12, 14, 16, 18\}.$$

Interested readers may verify (or extend!) this sequence. Some of these patterns for $n < 16$ can be formed with a subset of the seven Tangram tiles; all such are shown in Figure 4. Traditional Tangram puzzles require the use of all seven tiles. This is satisfied by exactly three *sets* of Figure 4 patterns: {a,c,d}, {a,i}, and {d,d}. These could be added to the cannon of Tangram puzzles. Members of such sets could be stacked on top of one another to form additional (albeit highly improper!) four-fold Tangrams. Similarly, four sets of Figure 4 patterns can be formed using all the Sei Shonagon Chie no Ita tiles: {a,h}, {a,j}, {b,f}, and {b,g}. Interested readers may similarly explore the sets of four-fold patterns that can be formed by the Fox-Epstein, Katsumata, and Uehara tiles.

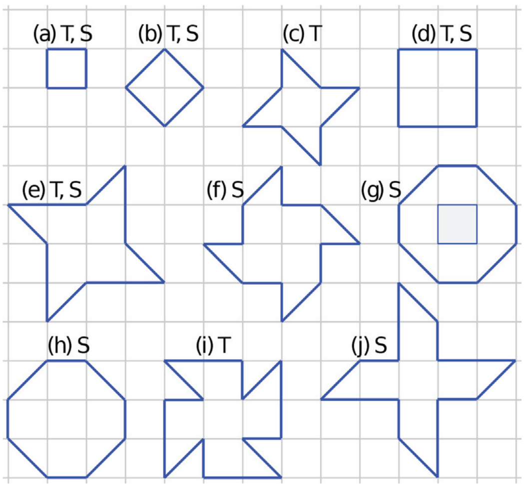

Figure 4 Four-fold polyabolos that can be formed by a subset of Tangram (T) and Sei Shonagon Chie no Ita (S) tiles. The shapes in the top row have size 2, 4, 6, and 8 left-to-right; the shapes in the middle and bottom rows have size 12 and 14 respectively.

Extensions

In closing, I would like to pose some further questions along the above lines. How many size-16 polyabolos exist with other kinds of symmetry, and how many of these can be formed by Tangram and Sei Shonagon Chie no Ita tiles? This would include patterns with one or two axes of reflection symmetry, and patterns with two-fold rotational symmetry. What alternative seven-piece dissections are optimal for constructing each category of pattern? How do the fractions of symmetric and constructible patterns vary with the number of base units in the "home" square (the possible square polyabolo sizes are two and four times an integer squared) and the number of tiles in the dissection? The Chinese Tangram is the most popular and well-known silhouette puzzle, but is this merely historical accident or is there some sense in which it represents an optimal dissection of the square?

Acknowledgments This work was supported by the National Science Foundation under Grant number DMR-1619625.

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Summary. It is proved by construction that only two Tangrams exist with four-fold rotational symmetry. One is the iconic simply-connected square, and the other is a relatively unknown chiral pinwheel with a central square hole. Both are size-16 polyabolos.

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The AM-GM Inequality: A Proof to Remember

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Perhaps, the most celebrated inequality of them all is the arithmetic mean-geometric mean (AM-GM) inequality. Let

$$\mathrm{AM} = \frac{a_1 + \cdots + a_n}{n},$$

and

$$\mathrm{GM} = \sqrt[n]{a_1 \cdots a_n}.$$

If $a_1, \ldots, a_n$ are nonnegative real numbers, then we have that AM $\geq$ GM. This inequality has been described as "the fundamental theorem of the theory of inequalities, the keystone on which many other important results rest" [2]. Many mathematicians find it pedagogically useful to present applications of the inequality. However, there seems to be just a small percentage of students who actually know how to prove it.

Of course, over the years many proofs of this result have been published (see Alzer [1], Cauchy [3], or Gwanyama [4], among many other examples), but many of them use sophisticated tools such as Cauchy's forward–backward induction or the method of Lagrange multipliers. The aim of this note is to supply a more flexible and easy-to-remember proof which does not seem to have appeared in the literature.

The following proof consists of only two main (and common) ingredients: induction and basic calculus.

Theorem. For all $a_1, \ldots, a_n \geq 0$, we have

$$\frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n} \tag{1}$$

Proof. We first consider the base cases. The theorem is trivially true for $n = 1$. If $n = 2$, then equation (1) is equivalent to

$$(\sqrt{a_1} - \sqrt{a_2})^2 \geq 0,$$

which is obviously true.

Suppose now that the theorem holds true for $n - 1$, that is

$$\frac{a_1 + \cdots + a_{n-1}}{n-1} \geq \sqrt[n-1]{a_1 \cdots a_{n-1}}, \tag{2}$$

for appropriate choices of $a_1, \ldots, a_{n-1}$. In order to show that (1) is true for n, we will use the following (crucial) trick: Set $a = a_1$. Then define the numbers $b_1, \ldots, b_{n-1}$ by

Math. Mag. **94** (2021) 302–303. doi:10.1080/0025570X.2021.1951581 © Mathematical Association of America
MSC: 97H30, 26D15

the formula $b_i = \frac{a_{i+1}}{a}$, with $1 \leq i \leq n-1$. With this notation, the AM − GM inequality is equivalent to

$$\frac{a \cdot (1 + b_1 + \cdots + b_{n-1})}{n} \geq \sqrt[n]{a^n \cdot b_1 \cdots b_{n-1}}.$$

We can cancel a from both sides and multiply by n. We now only need to show that

$$1 + b_1 + \cdots + b_{n-1} \geq n \cdot \sqrt[n]{b_1 \cdots b_{n-1}}$$

This can be done by manipulating the induction hypothesis (2) in the following way: We change a_i to b_i, $i = 1, \ldots, n-1$, multiply by $n-1$, and add 1 to both sides. We obtain that

$$1 + b_1 + \cdots + b_{n-1} \geq 1 + (n-1)\sqrt[n-1]{b_1 \cdots b_{n-1}}$$

holds true, so it suffices to show that

$$1 + (n-1)\sqrt[n-1]{b_1 \cdots b_{n-1}} \geq n\sqrt[n]{b_1 \cdots b_{n-1}}$$

If we set

$$x = \sqrt[n(n-1)]{b_1 \cdots b_{n-1}},$$

then the last inequality can be written in the form

$$1 + (n-1)x^n \geq nx^{n-1}$$

Finally, we let

$$f(x) = 1 + (n-1)x^n - nx^{n-1}, \ x \geq 0$$

and prove the last inequality using the first derivative test. We can see that

$$f'(x) = n(n-1)x^{n-2}(x-1).$$

It is an easy task to see that this function has a unique critical point, namely $x = 1$. Since $f'(x) > 0$ if $x > 1$, and $f'(x) < 0$ if $0 \leq x < 1$, the function has a global minimum at $x = 1$, the value $f(1) = 0$. This proves that $f(x) \geq 0$ and the inequality holds true for n. This completes the proof. ■

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Summary. We give a short and easy-to-remember proof of the arithmetic mean-geometric mean (AM-GM inequality), which does not seem to have appeared previously in the literature.

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Nancy Ho, James Godzik, Jennifer Jones, Thomas W. Mattman, and Dan Sours

"Invisible Knots and Rainbow Rings: Knots Not Determined by Their Determinants" MATHEMATICS MAGAZINE, Volume 93, Number 1, February 2020, pages 4-18.

This article has it all. Fun and games for all ages? Check. Careful, logical build-up of terminology and mathematical results? Check. A writing style that is conversational yet mathematically precise? Check. Fascinating connections across seemingly disparate areas of study within mathematics? Check. Well-chosen examples that answer reader questions just before we think to ask them? Check. Drawings to show those examples and build our intuition and understanding? Check. And finally: an ending that encourages the reader to do more, simultaneously issuing a challenge while also providing support for how to proceed? Check!

"Invisible Knots and Rainbow Rings: Knots Not Determined by Their Determinants" charts a delightful path that begins with Mobius strips, instantly engaging readers with ideas for Mobius strip variations to try themselves. We quickly learn these extensions are called paradromic rings, and there are patterns based on how many half twists we make, and based on whether we bisect the strip or cut it into even more sections. There is vocabulary for all this, starting with terms like knots and links, then describing the intriguingly named invisible knots and rainbow rings of the article's title.

Paradromic ring diagrams may be colored, similar to how graphs may be colored, and this is where the mathematical details and connections in this paper shine. Coloring requirements are first stated as two seemingly simple conditions, and the diagrams for some of the easier-to-visualize cases suggest it may be straightforward to decide colorability. However, readers likely guess that more complicated cases exist! This article talks us through these cases by providing multiple ways to determine the colorability of paradromic rings. We learn that drawings, besides building our intuition, also show a consistent way to re-draw paradromic rings to better visualize and count all their crossings. We revisit the single equation of the original two simple conditions, expanding this equation into a matrix–vector equation that is generalizable to examples involving many crossings. The eigenvalues of this matrix form the crux of proofs about colorability possibilities. We then learn about torus links, which partition paradromic rings into cases: some are torus links, and some are not. The authors build their case engagingly and convincingly, using all these ideas, and culminating with a complete characterization of the colorability of paradromic rings. Though they have proved all their results, they leave one proof out of the article as a temptation for readers. We are left with an invigorating call to learn more and complete the proof, as well as with

Math. Mag. **94** (2021) 304–307. doi:10.1080/0025570X.2021.1952041 © Mathematical Association of America
MSC: Primary 01A70

specific suggestions for books to read and topics to focus on. With this guidance, we as readers believe fully that we can progress on this work.

The authors have introduced us to their topic in a way that feels natural to anyone who has ever played with a Mobius strip. They then lead us through coloring, equations, knots, linear algebra, and proofs, all while sounding like we are chatting with a friend. They leave us inspired to try more and persuaded that we can very definitely make progress. Throughout, we remain engaged and find new connections in mathematics.

Response from the Authors

It is an honor to receive the Carl B. Allendoerfer award. Getting this paper published was a long and winding road and we're grateful to the many people who helped along the way. Thomas would like to dedicate this award to the many students who worked with him, especially those who worked hard only to find that the results could not be published. We also want to encourage those who have something to say to not be discouraged and to keep plugging away at it. In the hopes that it can be an inspiration, let us tell you that it took more than a decade between doing this research and getting it published.

This paper grew out of an REUT (Research Experience for Undergraduates and Teachers) at California State University, Chico that was supported in part by National Science Foundation REU Award 0354174, and by the Mathematical Association of America's NREUP program with funding from the NSF, NSA, and Moody's.

The first three authors were undergraduates at the time of this research, while Dan Sours was a high school teacher. We are grateful to Yuichi Handa, Ramin Naimi, Neil Portnoy, Robin Soloway, and John Thoo for helpful comments on early versions of this paper.

Additional funding came from California State University, Chico's Center for Excellence in Learning and Teaching (CELT) as part of a Faculty Learning Community. We thank Chris Fosen, Greg Cootsona, and the other FLC participants for fruitful discussions about the exposition.

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JENNIFER JONES (MR Author ID: 1361105) was an undergraduate at the Colorado State University, Fort Collins at the time of this research.

THOMAS W. MATTMAN (MR Author ID: 609682) received a PhD in Mathematics from McGill University, Canada. His mathematical interests include knot theory and graph theory and he enjoys supervising research by undergraduate students and teachers. In real life, he's a devoted husband to Shigemi and a doting father to Saya and Aki, who provide the comic relief.

DANIEL M. SOURS (MR Author ID: 1361267) received BS degrees in Mathematics (1985) and Engineering (1981) and an MS in Mathematics Education (2004) from California State University, Chico. He has taught at Chico High School in Chico, California since 1987 and has also served as adjunct faculty at the California State University, Chico and Butte Community College. He adores his wonderful wife Mary and loves relaxing with her on the ocean in Little River, California.

Jocelyn R. Bell and Frank Wattenberg,

"The Slippery Duck Theorem" Mathematics Magazine Volume 93, Number 2, April 2020, pages 91–103.

From the compelling title to the surprising and satisfying results, this article is a joy to read. The authors begin with a short history of the dog-and-duck problem, which has given rise to the study of pursuit curves in $\mathbb{R}^2$. Imagine a duck paddling along the edge of a pond with a given shape while a dog at the center of the pond starts making its way toward the duck. The dog's path is an example of a pursuit curve. Bell and Wattenberg put an interesting twist on this classic problem. They assume that the dog is swimming at a constant speed *slower* than the duck, and that the duck is "slippery", that is, it consistently gets away at the moment the dog captures the duck. By fixing the duck's path in advance, these assumptions produce some surprising results on the limiting behavior of the pursuit curve. The authors rely primarily on results from a standard first course in differential equations and a little analysis, together with Carathéodory's existence theorem and Brouwer's fixed-point theorem. With these tools, the authors prove (1) the existence of a cycle in the pursuit curve, (2) that such a cycle is unique, and (3) that such a cycle is independent of the dog's starting position (that is, a limit cycle). The proofs are clear, and the authors' use of various common tools, together with short explanations of less-commonly-taught results, makes this an excellent article for undergraduate math majors interested in exploring the next step beyond their differential equations class. In addition, the artistically pleasing examples provided by the authors suggest a plethora of "tweaks" that faculty might make to this particular problem to develop projects for their own students to consider. In this way, anyone reading the work will find something to intrigue and inspire.

Response from the Authors

We are absolutely delighted that our slippery duck paper has been selected for a Carl B. Allendoerfer award! We were searching for ways to include and engage cadets at West Point in mathematical exploration, and Hathaway's classic dog-and-duck problem from the Monthly fit the bill perfectly. We were ourselves surprised by the generality of our main result, the "slippery duck theorem." As an application of Brower's fixed point theorem, it is a nice reminder that abstract theorems in fields like topology sometimes have practical applications. We really had fun working on this problem, especially investigating limit cycles for different "duck paths." We used Mathematica, but any software that supports graphics and numerical solutions of systems of differential equations should work. This is a really rich source of student projects. There is a lot left to discover.

JOCELYN BELL (MR Author ID: 1057578) received her Ph.D. in 2011 from the State University of New York, with a concentration in general topology. From 2011 to 2016, she held a postdoctoral position at West Point, where she also worked on problems in network science. Since 2016 she has been an assistant professor in the department of mathematics at Hobart and William Smith Colleges. She has three little girls who love "playing numbers."

FRANK WATTENBERG (MR Author ID: 237940) retired at the end of June 2020 after over 50 years as a mathematician and mathematics teacher primarily interested in mathematical modeling for personal and public policy decisions. Like many of us, he has been forced by recent events to question the assumption that good science and good science education by themselves empower us to improve our world. The work of Dan Kahan and others at the Cultural Cognition Project is particularly important. Frank is hard at work on:

Seeing Stories through Everyday Cellphone Photography

Most of our students always have with them cellphones with remarkably capable cameras. **Seeing Stories** is about developing our students' powers of visual expression and narrative and about developing lifelong habits of everyday visual dairying. Visual narrative can nurture our sense of self and place in history and foster understanding and empathy across cultural divides.

Seeing Stories can become an engaging theme throughout students' academic lives—appearing in units from 15-minute units on "pictures of the day" from the morning's newspapers or Twitter feed to open-ended personal and creative projects. Along the way, students will develop standard material from the STEAM disciplines. As one example, middle school geometry and understanding of proportion and ratios is essential to composing effective photographs.

Seeing Stories is an example of the synergy between the sciences and the arts—for example, da Vinci's study of anatomy as he sought to capture people and animals on canvas—and modern digital image processing powering the creation of new mixed realities. **Seeing Stories** also multiplies the individual powers of images and words. Finally, and perhaps most importantly, students developing their powers of visual and verbal narrative do so by an ascending double helix of intertwined art appreciation and art creation.

PROBLEMS

Proposals

To be considered for publication, solutions should be received by March 1, 2022.

2126. *Proposed by M. V. Channakeshava, Bengaluru, India.*

A tangent line to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

meets the x-axis and y-axis at the points A and B, respectively.
Find the minimum value of AB.

2127. *Proposed by Jeff Stuart, Pacific Lutheran University, Tacoma, WA and Roger Horn, Tampa, FL.*

Suppose that $A, B \in M_{n\times n}(\mathbb{C})$ such that $AB = A$ and $BA = B$. Show that

(a) A and B are idempotent and have the same null space.
(b) If $1 \le \operatorname{rank} A < n$, then there are infinitely many choices of B that satisfy the hypotheses.
(c) $A = B$ if and only if $A - I$ and $B - I$ have the same null space.

2128. *Proposed by George Stoica, Saint John, NB, Canada.*

Let $0 < a < b < 1$ and $\epsilon > 0$ be given. Prove the existence of positive integers m and n such that $(1 - b^m)^n < \epsilon$ and $(1 - a^m)^n > 1 - \epsilon$.

Math. Mag. **94** (2021) 308–318. doi:10.1080/0025570X.2021.1957340

We invite readers to submit original problems appealing to students and teachers of advanced undergraduate mathematics. Proposals must always be accompanied by a solution and any relevant bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. Submitted problems should not be under consideration for publication elsewhere.

Proposals and solutions should be written in a style appropriate for this MAGAZINE.

Authors of proposals and solutions should send their contributions using the Magazine's submissions system hosted at http://mathematicsmagazine.submittable.com. *More detailed instructions are available there. We encourage submissions in PDF format, ideally accompanied by* LaTeX *source. General inquiries to the editors should be sent to* mathmagproblems@maa.org.

2129. *Proposed by Vincent Coll and Daniel Conus, Lehigh University, Bethlehem, PA and Lee Whitt, San Diego, CA.*

Determine whether the following improper integrals are convergent or divergent.

(a) $\displaystyle\int_0^1 \exp\left(\sum_{k=0}^{\infty} x^{2^k}\right) dx$

(b) $\displaystyle\int_0^1 \exp\left(\sum_{k=0}^{\infty} x^{3^k}\right) dx$

2130. *Proposed by Florin Stanescu, Şerban Cioiculescu School, Găeşti, Romania.*

Given the acute triangle ABC, let D, E, and F be the feet of the altitudes from A, B, and C, respectively. Choose $P, R \in \overleftrightarrow{AB}$, $S, T \in \overleftrightarrow{BC}$, $Q, U \in \overleftrightarrow{AC}$ so that

$$D \in \overleftrightarrow{PQ},\ E \in \overleftrightarrow{RS},\ F \in \overleftrightarrow{TU} \text{ and } \overleftrightarrow{PQ} \parallel \overleftrightarrow{EF},\ \overleftrightarrow{RS} \parallel \overleftrightarrow{DF},\ \overleftrightarrow{TU} \parallel \overleftrightarrow{DE}.$$

Show that

$$\frac{PQ+RS-TU}{AB} + \frac{RS+TU-PQ}{BC} + \frac{TU+PQ-RS}{AC} = 2\sqrt{2}$$

if and only if the circumcenter of $\triangle ABC$ lies on the incircle of $\triangle ABC$.

Quickies

1113. *Proposed by Philippe Fondanaiche, Paris, France.*

A generic n-gon is a convex polygon in which no three diagonals meet at a point in the interior of the n-gon. Determine the total number of triangles lying in the interior of a generic n-gon all of whose sides lie on the diagonals or sides of the n-gon.

1114. *Proposed by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain.*

Let F_k denote the kth Fibonacci number defined by initial values $F_0 = 0$, $F_1 = 1$ and the recurrence relation $F_{k+2} = F_{k+1} + F_k$ for $k \geq 0$. Find the value of the sum

$$\sum_{k=2}^{\infty} \arctan \frac{F_{k-1}}{F_k F_{k+1} + 1} \arctan \frac{F_{k+2}}{F_k F_{k+1} - 1}.$$

Solutions

Invariance of a ratio of sums of cotangents **October 2020**

2101. *Proposed by Michael Goldenberg, The Ingenuity Project, Baltimore Polytechnic Institute, Baltimore MD and Mark Kaplan, Towson University, Towson, MD.*

Recall that the Steiner inellipse of a triangle is the unique ellipse that is tangent to each side of the triangle at the midpoints of those sides. Consider the Steiner inellipse E_S of $\triangle ABC$ and another ellipse, E_A, passing through the centroid G of $\triangle ABC$ and tangent to $\overleftrightarrow{AB}$ at B and to $\overleftrightarrow{AC}$ at C. If E_S and E_A meet at M and N, let $\angle MAN = \alpha$. Construct ellipses E_B and E_C, introduce their points of intersection with E_S, and define angles β and γ in an analogous way. Prove that

$$\frac{\cot\alpha + \cot\beta + \cot\gamma}{\cot A + \cot B + \cot C} = \frac{11}{3\sqrt{5}}.$$

Solution by Albert Stadler, Herrliberg, Switzerland.
We first consider the equilateral triangle with vertices

$$A = (16, 0),\ B = (-8, 8\sqrt{3}),\quad \text{and}\quad C = (-8, -8\sqrt{3}),$$

whose centroid is the origin. In this case, E_S is the circle whose equation is $x^2 + y^2 = 8^2$ and E_A is the circle whose equation is $(x+16)^2 + y^2 = 16^2$. Solving this system of equations we find

$$M = (-2, 2\sqrt{15}) \quad \text{and} \quad N = (-2, -2\sqrt{15}).$$

Let $\angle(\overrightarrow{u}, \overrightarrow{v})$ denote the angle between the vectors $\overrightarrow{u}$ and $\overrightarrow{v}$. Then

$$A = \angle\Big((-24, 8\sqrt{3}), (-24, -8\sqrt{3})\Big) \quad \text{and} \quad \alpha = \angle\Big((-18, 2\sqrt{15}), (-18, -2\sqrt{15})\Big).$$

Rotating the vectors above 120° and 240° counter-clockwise gives

$$B = \angle\Big((0, -16\sqrt{3}), (24, -8\sqrt{3})\Big),$$
$$\beta = \angle\Big((9 - 3\sqrt{5}, -9\sqrt{3} - \sqrt{15}), (9 + 3\sqrt{5}, -9\sqrt{3} + \sqrt{15})\Big),$$
$$C = \angle\Big((24, 8\sqrt{3}), (0, 16\sqrt{3})\Big), \quad \text{and}$$
$$\gamma = \angle\Big((9 + 3\sqrt{5}, 9\sqrt{3} - \sqrt{15})), (9 - 3\sqrt{5}, 9\sqrt{3} + \sqrt{15})\Big).$$

Now let $\triangle A'B'C'$ be any non-degenerate triangle whose centroid is at the origin. There is an invertible linear map $f(x, y) = (ax + by, cx + dy)$ such that $\triangle A'B'C' = f(\triangle ABC)$. This linear mapping preserves the centroid, all midpoints, all tangencies, and it maps lines to lines and circles to ellipses. It remains to analyze how this linear mapping transforms the six numbers $\cot A$, $\cot B$, $\cot C$, $\cot\alpha$, $\cot\beta$, and $\cot\gamma$ to $\cot A'$, $\cot B'$, $\cot C'$, $\cot\alpha'$, $\cot\beta'$, and $\cot\gamma'$.

We will use the fact if $\phi = \angle((u_1, u_2), (v_1, v_2))$, then

$$\cot\phi = \frac{u_1v_1 + u_2v_2}{u_1v_2 - u_2v_1}$$

by the difference formula for cotangent.

Now

$$A' = \angle\Big(f(-24, 8\sqrt{3}), f(-24, -8\sqrt{3})\Big),$$
$$B' = \angle\Big(f(0, -16\sqrt{3}), f(24, -8\sqrt{3})\Big), \quad \text{and}$$
$$C' = \angle\Big(f(24, 8\sqrt{3}), f(0, 16\sqrt{3})\Big).$$

This gives

$$\cot A' = \frac{3a^2 - b^2 + 3c^2 - d^2}{2\sqrt{3}(ad - bc)}$$

$$\cot B' = \frac{b^2 - \sqrt{3}ab + d^2 - \sqrt{3}cd}{\sqrt{3}(ad - bc)}$$

$$\cot C' = \frac{b^2 + \sqrt{3}ab + d^2 + \sqrt{3}cd}{\sqrt{3}(ad - bc)}.$$

Therefore,

$$\cot A' + \cot B' + \cot C' = \frac{\sqrt{3}\left(a^2 + b^2 + c^2 + d^2\right)}{2(ad - bc)}.$$

A similar calculation yields

$$\cot \alpha' + \cot \beta' + \cot \gamma' = \frac{11\left(a^2 + b^2 + c^2 + d^2\right)}{2\sqrt{15}(ad - bc)}.$$

Finally,

$$\frac{\cot \alpha' + \cot \beta' + \cot \gamma'}{\cot A' + \cot B' + \cot C'} = \frac{11}{3\sqrt{5}}$$

as desired.

Also solved by Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia) and the proposers. There were two incomplete or incorrect solutions.

Trigonometric identities for the heptagonal triangle **October 2020**

2102. *Proposed by Donald Jay Moore, Wichita, KS.*

Let $\alpha = \pi/7$, $\beta = 2\pi/7$, and $\gamma = 4\pi/7$. Prove the following trigonometric identities.

$$\frac{\cos^2 \alpha}{\cos^2 \beta} + \frac{\cos^2 \beta}{\cos^2 \gamma} + \frac{\cos^2 \gamma}{\cos^2 \alpha} = 10,$$

$$\frac{\sin^2 \alpha}{\sin^2 \beta} + \frac{\sin^2 \beta}{\sin^2 \gamma} + \frac{\sin^2 \gamma}{\sin^2 \alpha} = 6,$$

$$\frac{\tan^2 \alpha}{\tan^2 \beta} + \frac{\tan^2 \beta}{\tan^2 \gamma} + \frac{\tan^2 \gamma}{\tan^2 \alpha} = 83.$$

Solution by Eugene A. Herman, Grinnell College, Grinnell, IA.
Denote the trigonometric expressions by $\mathcal{C}, \mathcal{S}, \mathcal{T}$, respectively. The expansion

$$\sin(7t) = \sin t \left(64\cos^6 t - 80\cos^4 t + 24\cos^2 t - 1\right)$$

yields the key polynomial as follows. When $t = \alpha$ or $t = \beta$ or $t = \gamma$, then $\sin(7t) = 0$ but $\sin t \neq 0$. Hence the cubic polynomial

$$p(x) = 64x^3 - 80x^2 + 24x - 1$$

has the three zeros $a = \cos^2\alpha, b = \cos^2\beta, c = \cos^2\gamma$. Since

$$p(x) = 64(x-a)(x-b)(x-c),$$

we have values for the three elementary symmetric polynomials:

$$a+b+c = \frac{5}{4}, \quad ab+bc+ca = \frac{3}{8}, \quad abc = \frac{1}{64}.$$

We use the double angle formula for sine as follows:

$$\frac{\sin^2 t}{\sin^2 2t} = \frac{\sin^2 t}{4\sin^2 t\cos^2 t} = \frac{1}{4\cos^2 t}.$$

Hence, since $\sin^2 2\gamma = \sin^2\alpha$,

$$\mathcal{S} = \frac{\sin^2\alpha}{\sin^2\beta} + \frac{\sin^2\beta}{\sin^2\gamma} + \frac{\sin^2\gamma}{\sin^2\alpha} = \frac{1}{4a} + \frac{1}{4b} + \frac{1}{4c} = \frac{bc+ca+ab}{4abc} = \frac{3/8}{4/64} = 6.$$

We use the double angle formula for cosine as follows:

$$\frac{\cos^2 t}{\cos^2 2t} = \frac{\cos^2 t}{(2\cos^2 t - 1)^2}.$$

Hence, since $\cos^2 2\gamma = \cos^2\alpha$,

$$\mathcal{C} = \frac{\cos^2\alpha}{\cos^2\beta} + \frac{\cos^2\beta}{\cos^2\gamma} + \frac{\cos^2\gamma}{\cos^2\alpha} = \frac{a}{(2a-1)^2} + \frac{b}{(2b-1)^2} + \frac{c}{(2c-1)^2}.$$

Substituting $x = (y+1)/2$ into the polynomial $p(x)$ yields

$$q(y) = 8y^3 + 4y^2 - 4y - 1.$$

Since $y = 2x-1$, the zeros of $q(y)$ are $a' = 2a-1, b' = 2b-1, c' = 2c-1$ and the elementary symmetric polynomial expressions are

$$a'+b'+c' = -\frac{1}{2}, \quad a'b'+b'c'+c'a' = -\frac{1}{2}, \quad a'b'c' = \frac{1}{8}.$$

Hence,

$$\mathcal{C} = \frac{a'+1}{2a'^2} + \frac{b'+1}{2b'^2} + \frac{c'+1}{2c'^2} = \frac{a'b'^2c'^2 + b'a'^2c'^2 + c'a'^2b'^2 + b'^2c'^2 + a'^1c'^2 + a'^2b'^2}{2(a'b'c')^2}$$

$$= \frac{(a'b'c')(a'b'+b'c'+c'a') + (a'b'+b'c'+c'a')^2 - 2(a'b'c')(a'+b'+c')}{2(a'b'c')^2}$$

$$= \frac{-1/16 + 1/4 + 1/8}{2/64} = 10.$$

For the third identity, we use both double angle formulas:

$$\frac{\tan^2 t}{\tan^2 2t} = \frac{\sin^2 t\ \cos^2 2t}{\cos^2 t\ \sin^2 2t} = \frac{(2\cos^2 t - 1)^2}{4\cos^4 t}$$

Thus, since $\tan^2 2\gamma = \tan^2\alpha$,

$$\mathcal{T} = \frac{\tan^2\alpha}{\tan^2\beta} + \frac{\tan^2\beta}{\tan^2\gamma} + \frac{\tan^2\gamma}{\tan^2\alpha} = \left(\frac{2a-1}{2a}\right)^2 + \left(\frac{2b-1}{2b}\right)^2 + \left(\frac{2c-1}{2c}\right)^2.$$

Substituting $x = 1/(2(1-z))$ into the polynomial $p(x)$ and clearing fractions yields

$$r(z) = 8(z^3 + 9z^2 - z - 1).$$

Since $z = (2x-1)/(2x)$, the zeros of $r(z)$ are

$$a' = \frac{2a-1}{2a}, \quad b' = \frac{2b-1}{b}, \quad c' = \frac{2c-1}{c}$$

and the elementary symmetric polynomial expressions are

$$a' + b' + c' = -9, \quad a'b' + b'c' + c'a' = -1, \quad a'b'c' = 1.$$

Hence,

$$\mathcal{T} = a'^2 + b'^2 + c'^2 = (a' + b' + c')^2 - 2(a'b' + b'c' + c'a') = 9^2 - 2(-1) = 83.$$

Also solved by Michel Bataille (France), Anthony J. Bevelacqua, Brian Bradie, Robert Calcaterra, Hongwei Chen, John Christopher, Robert Doucette, Habib Y. Far, J. Chris Fisher, Dmitry Fleischman, Michael Goldenberg & Mark Kaplan, Russell Gordon, Walther Janous (Austria), Kee-Wai Lau (Hong Kong), James Magliano, Ivan Retamoso, Volkhard Schindler (Germany), Randy Schwartz, Allen J.Schwenk, Albert Stadler (Switzerland), Seán M. Stewart (Australia), Enrique Treviño, Michael Vowe (Switzerland), Edward White & Roberta White, Lienhard Wimmer (Germany), and the proposer. There were two incomplete or incorrect solutions.

How many tickets to buy to guarantee three out of four? October 2020

2103. *Proposed by Péter Kórus, University of Szeged, Szeged, Hungary.*

In a soccer game there are three possible outcomes: a win for the home team (denoted 1), a draw (denoted X), or a win for the visiting team (denoted 2). If there are n games, betting slips are printed for all 3^n possible outcomes. For four games, what is the minimum number of slips you must purchase to guarantee that at least three of the outcomes are correct on at least one of your slips?

Solution by Northwestern University Math Problem Solving Group, Northwestern University, Evanston, IL.

The answer is nine.

First, we prove that it is impossible to guarantee at least three correct outcomes with fewer than nine slips.

Let T be the set of all possible outcomes, i.e., all 4-tuples of 1, X, and 2. There are $3^4 = 81$ such 4-tuples. In that set, we define the Hamming distance d as the number of places in which two tuples differ. For example, $d(1X21, 2X12) = 3$ because 1X21 and 2X12 differ in three places, namely the first, third and fourth places. The Hamming distance satisfies the usual axioms for a metric, and we can define balls in T in the usual way, i.e., a ball with center $c \in T$ and radius $r \in \mathbb{R}$ is

$$B_r(c) = \{t \in T \mid d(t, c) \leq r\}.$$

Given a tuple $c \in T$, the set of tuples that coincide with c in at least three places consists of those that differ from c in no more than one place. In other words, this set is $B_1(c)$. Note that $B_1(c)$ contains exactly 9 elements: the center c, the two tuples that differ from c exactly in the first element, the two that differ in the second, the two that differ in the third, and the two that differ in the fourth.

In order to ensure that our slips $c_1, c_2, \ldots, c_n$ contain at least three correct entries, the balls $B_1(c_i)$, $i = 1, 2, \ldots, n$ must cover T, i.e.,

$$T = \bigcup_{i=1}^{n} B_1(c_i).$$

Since $|B_1(c)| = 9$ and $|T| = 81$, we will need at least $81/9 = 9$ slips.

Next, we will prove that nine slips suffice. That can be accomplished by exhibiting nine 4-tuples $c_1, \ldots, c_9$ such that $B_i(c_1), \ldots, B_i(c_9)$ cover T, i.e., such that every element in T has a Hamming distance of at most 1 from at least one of the c_i. The following 4-tuples satisfy the condition:

1111 1XXX 1222 X1X2 XX21 X21X 2X12 212X 22X1

One (somewhat tedious) way to check it is to verify that each of the 81 elements in T differ from at least one of these tuples in no more one place.

A slightly easier way to verify the assertion is to observe that these tuples differ from each other in exactly three places, so the Hamming distance between any two of them is 3. Because of the triangle inequality, it is impossible for balls of radius 1 centered on the c_i to overlap. Therefore the total number of elements contained in the union of these balls is $9 \cdot 9 = 81$, so the union must be all of T.

This completes the proof.

Also solved by Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), Eagle Problem Solvers, Fresno State Problem Solving Group, Dan Hletko, Rob Pratt, Allen J. Schwenk, and the proposer. There were seven incomplete or incorrect solutions.

Vector spaces as unions of proper subspaces **October 2020**

2104. *Proposed by the Missouri State University Problem Solving Group, Missouri State University, Springfield, MO.*

It is well known that no vector space can be written as the union of two proper subspaces. For which m does there exist a vector space V that can be written as a union of m proper subspaces with this collection of subspaces being minimal in the sense that no union of a proper subcollection is equal to V?

Solution by Paul Budney, Sunderland, MA.

Such a decomposition exists for any $m > 2$.

Let $V = \mathbb{F}_2^n$, where $\mathbb{F}_2$ is the field with two elements. Let

$$V_i = \{(x_1, \ldots, x_n) \in V \mid x_i = 0\}$$

for $1 \le i \le n$ and let

$$W = \{(0, 0, \ldots, 0), (1, 1, \ldots, 1)\}.$$

Clearly W and the V_i are proper subspaces of V. Since $(1, 1, \ldots, 1)$ is the only vector not in $V_1 \cup V_2 \cup \ldots \cup V_n$,

$$W \cup V_1 \cup V_2 \cup \ldots \cup V_n = V.$$

Deleting W from this union excludes $(1, 1, \ldots, 1)$. Deleting V_i from this union excludes $(1, \ldots, 1, 0, 1, \ldots, 1)$, with 0 for the ith component and 1's elsewhere. Thus, there is no proper subcollection of these subspaces whose union is V. There are

$n+1$ subspaces, and since $n \geq 2$ is arbitrary, the desired decomposition exists for any $m > 2$.

Also solved by Anthony Bevelacqua, Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), Robert Doucette, Eugene Herman, and the proposer. There was one incomplete or incorrect solution.

An asymptotic formula for a definite integral **October 2020**

2105. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu", Bîrlad, Romania.*

Let $f : [0, 1] \to \mathbb{R}$ be a function that is k times differentiable on $[0, 1]$, with the kth derivative integrable on $[0, 1]$ and (left) continuous at 1. For integers $i \geq 1$ and $j \geq 0$ let

$$\sigma_j^{(i)} = \sum_{j_1+j_2\cdots+j_i=j} 1^{j_1} 2^{j_2} \cdots i^{j_i},$$

where the sum is extended over all i-tuples $(j_1, \ldots, j_i)$ of nonnegative integers that sum to j. Thus, for example, $\sigma_0^{(i)} = 1$, and $\sigma_1^{(i)} = 1 + 2 + \cdots + i = i(i+1)/2$ for all $i \geq 1$. Also, for $0 \leq j \leq k$ let

$$a_j = \sigma_j^{(1)} f(1) + \sigma_{j-1}^{(2)} f'(1) + \cdots + \sigma_1^{(j)} f^{(j-1)}(1) + \sigma_0^{(j+1)} f^{(j)}(1).$$

Prove that

$$\int_0^1 x^n f(x)dx = \frac{a_0}{n} - \frac{a_1}{n^2} + \cdots + (-1)^k \frac{a_k}{n^{k+1}} + o\left(\frac{1}{n^{k+1}}\right),$$

for $n \to \infty$. As usual, we denote by $f^{(s)}$ the sth derivative of f (with $f^{(0)} = f$), and by $o(x_n)$ a sequence (y_n) with the property that $\lim_{n\to\infty} y_n/x_n = 0$.

Solution by Michel Bataille, Rouen, France.
For $x \in [0, 1]$, let $f_0(x) = f(x)$ and

$$f_j(x) = \frac{d}{dx}\left(x f_{j-1}(x)\right), 1 \leq j \leq k.$$

An easy induction shows that for $0 \leq j \leq k$, the function f_j is a linear combination of the functions $f(x), xf'(x), \ldots, x^j f^{(j)}(x)$. It follows that $f_0, f_1, \ldots, f_{k-1}$ are differentiable on $[0, 1]$ and that f_k is integrable on $[0, 1]$ and continuous at 1.

Integrating by parts, we obtain the following recursion that holds for $1 \leq j \leq k-1$:

$$\int_0^1 x^n f_{j-1}(x)\, dx = \left[\frac{x^n}{n} \cdot (x f_{j-1}(x))\right]_0^1 - \frac{1}{n}\int_0^1 x^n f_j(x)\, dx$$
$$= \frac{f_{j-1}(1)}{n} - \frac{1}{n}\int_0^1 x^n f_j(x)\, dx.$$

With the help of this recursion, we are readily led to

$$\int_0^1 x^n f(x)\, dx = \int_0^1 x^n f_0(x)\, dx$$

$$= \sum_{j=0}^{k-1} (-1)^j \frac{f_j(1)}{n^{j+1}} + \frac{(-1)^k}{n^k} \int_0^1 x^n f_k(x)\, dx.$$

Now, if $g : [0, 1] \to \mathbb{R}$ is integrable on $[0, 1]$ and continuous at 1, then

$$\lim_{n\to\infty} n \cdot \int_0^1 x^n g(x)\, dx = g(1)$$

(Paulo Ney de Souza, Jorge-Nuno Silva, *Berkeley Problems in Mathematics*, Springer, 2004, Problem 1.2.13). With $g = f_k$, this yields

$$\int_0^1 x^n f_k(x)\, dx = \frac{f_k(1)}{n} + o\left(\frac{1}{n}\right)$$

and therefore

$$\int_0^1 x^n f(x)\, dx = \sum_{j=0}^{k-1} (-1)^j \frac{f_j(1)}{n^{j+1}} + \frac{(-1)^k}{n^k} \left(\frac{f_k(1)}{n} + o\left(\frac{1}{n}\right)\right)$$

$$= \sum_{j=0}^{k} (-1)^j \frac{f_j(1)}{n^{j+1}} + o\left(\frac{1}{n^{k+1}}\right).$$

Comparing this with the statement of the problem, it remains to prove that $a_j = f_j(1)$ for $0 \le j \le k$. Clearly, it is sufficient to prove that for $x \in [0, 1]$

$$f_j(x) = \sum_{i=0}^{j} \sigma_{j-i}^{(i+1)} x^i f^{(i)}(x). \tag{E_j}$$

We use induction. Since $f_0(x) = f(x) = 1 \cdot x^0 f^{(0)}(x)$, (E_0) holds. Before addressing the induction step, we establish two results about the numbers $\sigma_j^{(i)}$. The first result is

$$\sigma_j^{(i+1)} = \sum_{r=0}^{j} (1+i)^r \sigma_{j-r}^{(i)}. \tag{1}$$

Proof. When $j_1 + \cdots + j_i + j_{i+1} = j$, then j_{i+1} can take the values $0, 1, \ldots, j$. It follows that

$$\sigma_j^{(i+1)} = \sum_{j_1+\cdots+j_{i+1}=j} 1^{j_1} 2^{j_2} \cdots i^{j_i} (i+1)^{j_{i+1}}$$

$$= \sum_{r=0}^{j} (1+i)^r \sum_{j_1+\cdots+j_i=j-r} 1^{j_1} 2^{j_2} \cdots i^{j_i}$$

$$= \sum_{r=0}^{j} (1+i)^r \sigma_{j-r}^{(i)}.$$

The second result is

$$\sigma_{j+1}^{(i+1)} = \sigma_{j+1}^{(i)} + (1+i)\sigma_j^{(i+1)}. \tag{2}$$

Proof. Applying (1),

$$\begin{aligned}\sigma_{j+1}^{(i+1)} &= \sum_{r=0}^{j+1}(1+i)^r\sigma_{j+1-r}^{(i)} \\ &= \sigma_{j+1}^{(i)} + (1+i)\sum_{r=1}^{j+1}(1+i)^{r-1}\sigma_{j-(r-1)}^{(i)} \\ &= \sigma_{j+1}^{(i)} + (1+i)\sum_{r=0}^{j}(1+i)^r\sigma_{j-r}^{(i)}\end{aligned}$$

and applying (1) again we conclude that $\sigma_{j+1}^{(i+1)} = \sigma_{j+1}^{(i)} + (1+i)\sigma_j^{(i+1)}$.

Now, assume that (E_j) holds for some integer j such that $0 \le j \le k-1$. Then, we calculate

$$\begin{aligned}f_{j+1}(x) &= \frac{d}{dx}\left[\sum_{i=0}^{j}\sigma_{j-i}^{(i+1)}x^{i+1}f^{(i)}(x)\right] \\ &= \sum_{i=0}^{j}\sigma_{j-i}^{(i+1)}(i+1)x^i f^{(i)}(x) + \sum_{i=0}^{j}\sigma_{j-i}^{(i+1)}x^{i+1}f^{(i+1)}(x) \\ &= \sum_{i=0}^{j}\sigma_{j-i}^{(i+1)}(i+1)x^i f^{(i)}(x) + \sum_{i=1}^{j+1}\sigma_{j-i+1}^{(i)}x^i f^{(i)}(x) \\ &= \sigma_j^{(1)}f(x) + \sum_{i=1}^{j}\Big([\sigma_{j-i+1}^{(i)} + (i+1)\sigma_{j-i}^{(i+1)}]x^i f^{(i)}(x)\Big) + \sigma_0^{(j+1)}x^{j+1}f^{(j+1)}(x).\end{aligned}$$

Using (2) and $\sigma_j^{(1)} = \sigma_{j+1}^{(1)} = 1 = \sigma_0^{(j+1)} = \sigma_0^{(j+2)}$, we see that

$$f_{j+1}(x) = \sum_{i=0}^{j+1}\sigma_{j+1-i}^{(i+1)}x^i f^{(i)}(x)$$

so that (E_{j+1}) holds. This completes the induction step and the proof.

Note. The number $\sigma_j^{(i)}$ is the Stirling number of the second kind $S(i+j, i) = \left\{ {i+j \atop i} \right\}$ (see L. Comtet, *Advanced Combinatorics*, Reidel, 1974, Theorem D p. 207).

Also solved by Albert Stadler (Switzerland) and the proposer.

Answers

Solutions to the Quickies from page 309.

A1113. Consider the union of the endpoints of the sides or diagonals of the polygon that contain the sides of the interior triangle. There can be 3,4,5, or 6 such points. Order those points in a clockwise direction around the polygon: $P_1, P_2, \ldots$.

- With three points, the sides of the triangle must be $\{P_1P_2, P_1P_3, P_2P_3\}$. There are $\binom{n}{3}$ of these triangles.

- With four points, there are four possibilities for the sides of the triangle: $\{\{P_i P_{i+2},\ P_{i+2}P_{i+1},\ P_{i+1}P_{i+3}\}\}$, where the subscripts are taken modulo 4. There are $4\binom{n}{4}$ of these triangles.
- With five points, there are five possibilities for the sides of the triangle: $\{\{P_i P_{i+2},\ P_{i+2}P_{i+4},\ P_{i+1}P_{i+3}\}\}$, where the subscripts are taken modulo 5. There are $5\binom{n}{5}$ of these triangles.
- With six points, the sides of the triangle must be $\{P_1P_4,\ P_2P_5,\ P_3P_6\}$. There are $\binom{n}{6}$ of these triangles.

Therefore, the total number of triangles is

$$\binom{n}{3} + 4\binom{n}{4} + 5\binom{n}{5} + \binom{n}{6}.$$

A1114. Let

$$S_n = \sum_{k=2}^{n} \arctan \frac{F_{k-1}}{F_k F_{k+1} + 1} \arctan \frac{F_{k+2}}{F_k F_{k+1} - 1}.$$

By the recurrence relation for the Fibonacci numbers, we have

$$S_n = \sum_{k=2}^{n} \arctan \frac{F_{k+1} - F_k}{F_k F_{k+1} + 1} \arctan \frac{F_k + F_{k+1}}{F_k F_{k+1} - 1}.$$

Since,

$$\arctan \frac{y-x}{xy+1} = \arctan \frac{1}{x} - \arctan \frac{1}{y} \quad \text{and} \quad \arctan \frac{y+x}{xy-1} = \arctan \frac{1}{x} + \arctan \frac{1}{y}$$

for $xy > 1$, our sum becomes

$$\begin{aligned} S_n &= \sum_{k=2}^{n} \left(\arctan \frac{1}{F_k} - \arctan \frac{1}{F_{k+1}}\right)\left(\arctan \frac{1}{F_k} + \arctan \frac{1}{F_{k+1}}\right) \\ &= \sum_{k=2}^{n} \left(\arctan \frac{1}{F_k}\right)^2 - \left(\arctan \frac{1}{F_{k+1}}\right)^2 \\ &= \left(\arctan \frac{1}{F_2}\right)^2 - \left(\arctan \frac{1}{F_{n+1}}\right)^2, \end{aligned}$$

since the last sum telescopes. Hence, the sum of the series is

$$\begin{aligned} \lim_{n\to\infty} S_n &= \lim_{n\to\infty} \left(\left(\arctan \frac{1}{F_2}\right)^2 - \left(\arctan \frac{1}{F_{n+1}}\right)^2\right) \\ &= \left(\arctan \frac{1}{F_2}\right)^2 = \frac{\pi^2}{16}. \end{aligned}$$

REVIEWS

PAUL J. CAMPBELL, *Editor*
Beloit College

Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles, books, and other materials are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of mathematics literature. Readers are invited to suggest items for review to the editors.

Beeson, Michael, Euclid after computer proof-checking, https://arxiv.org/pdf/2103.09623.pdf.

In my February column (94 (2021) 79), I drew attention to Stephen Wolfram's graphs showing the interdependence of the theorems, definitions, and axioms in Euclid. The article at hand takes a different approach, analyzing and fixing various omissions and oversights in Euclid. The author and colleagues translated the definitions, axioms, and propositions in Euclid into computer proof-checkers and produced formal proofs of the propositions, after judiciously changing and adding axioms (collinearity, betweenness, and more). One notable result is the correction of a score of proofs of propositions in Book I; another is the machine generation of human-readable two-column proofs at a user-chosen level of detail. Anyone who teaches or plans to teach high school or college geometry would benefit from careful reading of this paper (as well as Wolfram's). Beeson and colleagues have realized the dream of Gerolamo Saccheri in his 1733 book *Euclid Freed of Every Flaw*.

Wood, Charlie, Imaginary numbers may be essential for describing reality, https://www.quantamagazine.org/imaginary-numbers-may-be-essential-for-describing-reality-20210303/.

Well, we may *really* need complex numbers. "In quantum mechanics, the behavior of a particle or group of particles is encapsulated by a wavelike entity known as the wave function, or Ψ," whose behavior is described by the Schrödinger wave equation. That equation contains the imaginary unit i, but Schrödinger and subsequent physicists have tried to shoo the i away. A team of quantum theorists has designed an experiment about entanglement of particles whose outcome would be expected either to require a complex-number description or else to invalidate standard quantum mechanics theory. If the theory is correct—and most researchers are confident that it is—"then real numbers alone cannot fully describe nature."

Jungic, Veselin, Making calculus relevant: Final exam in the time of COVID-19, *International Journal of Mathematical Education in Science and Technology*, https://doi.org.10.1080/0020739X.2020.1775903.

The author used news reports and graphics about the spread of the pandemic to illustrate concepts in his course in differential calculus. He then composed questions for the final course exam that tested students' abilities to interpret a news graphic in calculus terms, do associated calculations, and offer both analytical and verbal interpretations. The article includes the graphic, the exam questions, their categorization in terms of Bloom's taxonomy, sample answers, and comments on the answers received.

Lievenlb [Le Bruyn, Lieven], The symmetries of Covid-19, http://www.neverendingbooks.org/the-symmetries-of-covid-19.

Illustrations of the SARS-CoV-2 virus, with the "spikes" (S-proteins) on its surface, tend to resemble stellated icosahedra (many other viruses do have that shape). But the virus on average has 74 such spikes, impossible for the icosahedral rotation group, which has order 60. However, octahedral symmetry cannot be ruled out.

Math. Mag. **94** (2021) 319–320. doi:10.1080/0025570X.2021.1957349

Bressoud, David, Thoughts on Advanced Placement Precalculus, https://www.mathvalues.org/masterblog/thoughts-on-advanced-placement-precalculus.

The College Board is exploring offering an AP Precalculus course, and author Bressoud is on the advisory board. “Precalculus is a high school course,” Bressoud asserts, but notes that an AP exam would take pressure off demand for AP Calculus and improve precalculus as taught in high school. AP Precalculus would not receive credit at many colleges and universities; however, many students take precalculus or college algebra as a dual enrollment course and thus receive both high school and college credit for it. A key consideration is the intent and purpose of high school mathematics: “[C]alculus dominates and drives the high school curriculum, even and especially for those students who have no desire to ever study it.” AP Calculus serves mainly for college admissions enhancement—most AP Calculus students take no further mathematics in college. Many years ago, my institution, Beloit College, stopped offering liberal arts math/science distribution credit for precalculus because of its singular focus on technique; thereafter, enrolment plummeted, very few students from the course completed any calculus, and the course was abandoned. Later, a brief experiment offered just-in-time extra precalculus sessions in parallel with calculus; the experiment was judged unsuccessful. The College recently re-instituted a precalculus course to address needs of a greater proportion of underprepared students seeking STEM careers. Meanwhile—last year 50,000 students took the BC calculus exam *as juniors*, leading to pressure for an AP Multivariable Calculus course.

Houston-Edwards, Kelsey, The math of making connnections, *Scientific American* 324 (4) (April 2021), 22–29.

This article discusses percolation theory, which considers properties of networks based on their degree of connectivity. Applications vary from oil flowing through porous rock to water filtering through coffee. A common phenomenon is a phase transition at a threshold of connectivity; for example, fracturing the rock can get oil to flow all the way through, or precautions can prevent the spread of a disease through an entire population. A network’s threshold is lower if there is a wide range of degrees of the nodes in the network, so a disease will spread more readily if some people are isolated but others are highly connected (“superspreaders”).

Euleriana. https://scholarlycommons.pacific.edu/euleriana/.

The first issue is out of *Euleriana*, a peer-reviewed “eJournal” devoted to Euler and Euler-related scholarship, including translations, commentaries, and articles, with all content freely available. Did you know that Euler wrote about the flow of blood through arteries? or had theories of musical tuning?

Hughes, G.H., The polygons of Albrecht Dürer—1525. https://arxiv.org/abs/1205.0080v1.

Albrech Dürer is known for his prints, but he also published a book on geometry, which included ruler-and-compass constructions for regular n-gons, for $n = 3, \ldots, 16$. This paper examines those constructions with Dürer’s explanations of them and then evaluates the constructions that are only approximate. His “trisection” of a 60° angle produces an angle of 19° 59’ 59.00005”.

Wilson, Glenn, Klaki is a tricky jigsaw app based on an ancient mathematical problem, out now on iOS, https://www.gamezebo.com/2021/03/09/klaki-is-a-tricky-jigsaw-app-based-on-an-ancient-mathematical-problem-out-now-on-ios/.

“You can’t square the circle” is a way to say that a task is impossible, but “squaring the circle” originally referred to constructing a square with the same area as a given circle using a finite number of steps with compass and straightedge (Lindemannn in 1882 showed that it can’t be done). A related problem, due to Tarski in 1925, is to dissect a circular disk into a finite number of pieces and reassemble those into a square. Ezekiel Thekiso has written a free cellphone app that offers the opportunity to choose a number of pieces, decide whether to build them into a disk or into a square, and try to do so (with hints provided).